CS243: Discrete Structures

Sets

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Announcements

- Third homework is out
- Second homework is due now
- First homework is graded ⇒ handed back at end of lecture
- Scores posted on Blackboard – check your score!
  - Have one week to report any inconsistencies between Blackboard and your actual score
  - Don't tell this to us at end of semester

Introduction

- Sets are the most basic, fundamental data structures in math and computer science
- Many of you should be familiar with sets from high school
- Partly review to refresh your memory
- Partly application of proof techniques to sets

Sets and Set Membership

- A set is unordered collection of objects
- Example: Vowels in the English language: { a, e, i, o, u }
- Example: Positive even numbers less than 10: {2, 4, 6, 8}
- Objects in set S are called members (or elements) of that set
- If x is a member of S, we write $x \in S$

Cardinality

- Number of elements in a set is called its cardinality
- What is cardinality of $\{a, e, i, o, u\}$?
- Cardinality of a set S is written as $|S|$
- Sets we looked at so far only contain finitely many elements, i.e., have finite cardinality
- But in general sets can have infinite cardinality
- Example: Sets of all natural numbers: $\mathbb{N} := \{0, 1, 2, 3, ...\}$

Important Sets in Mathematics

- Many sets that play fundamental role in mathematics have infinite cardinality
- Set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Set of positive integers: $\mathbb{Z}^+ = \{1, 2, \ldots\}$
- Set of real numbers: $\mathbb{R} = \{\pi, \ldots, -1.999, \ldots, 0, \ldots, 0.000001, \ldots\}$
- Infinite sets are easier to write in set builder notation

Set Builder Notation

- Set builder notation describes a set S by stating a property all elements of S must have, written:

$$S = \{x \mid x \text{ has property } p\}$$

- This means S consists of all elements that have property p
- Alternatively, an object x is member of S iff x has property p
- Example: $S = \{x \mid x \in \mathbb{Z} \ \wedge \ x\%2 = 0\}$
- Example: $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z} \ \wedge \ q \in \mathbb{Z} \ \wedge \ q \neq 0\}$
- $\mathbb{Q}$ is the set of rational numbers

Set Equality

- Two sets A and B are equal iff they have the same elements
- If A and B are equal, we write $A = B$
- Are the sets $\{a, b, c\}$ and $\{b, a, c\}$ equal?
- Are the sets $\mathbb{N}$ and $\mathbb{Z}^+$ equal?
- Are the sets $\{1, 2, 3\}$ and $\{1, 2, 2, 3\}$ equal?

Special Sets

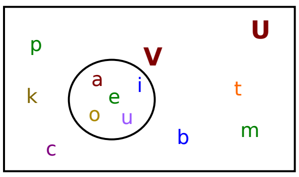
- The universal set, written U, includes all objects under consideration
- The empty set, written $\emptyset$ or $\{\}$, contains no objects
- A set containing exacly one element is called a singleton set

Questions about Special Sets

- What special set is $S = \{x \mid p(x) \wedge \neg p(x)\}$ equal to?
- What is the cardinality of a singleton set?
- What is the cardinality of $\mathbb{R}$?
- What is the cardinality of the set $\emptyset$?
- What is the cardinality of $\{\emptyset\}$?

Venn Diagrams

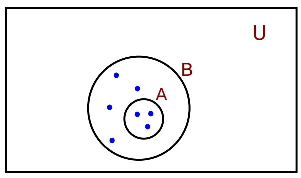
- Sometimes convenient to visualize sets using Venn diagrams
- Example: Venn diagram for vowels in English alphabet



- Since objects under consideration are letters in English alphabet, U contains all 26 letters

Subsets

- A set A is a subset of set B, written $A \subseteq B$, iff every element in A is also an element of B ($\forall x.\ x \in A \Rightarrow x \in B$)



- For any set A, we have $A \subseteq A$.
- For any set A, $\emptyset \subseteq A$.

Supersets and Proper Subsets

- If $A \subseteq B$, then B is called a superset of A, written $B \supseteq A$
- Example: For every set A, $U \supseteq A$
- Another important concept: proper subset
- A set A is a proper subset of set B, written $A \subset B$, iff:

$$(\forall x.\ x \in A \Rightarrow x \in B) \wedge (\exists x.\ x \in B \wedge x \notin A)$$

Questions about Subsets and Supersets

- Is there some set for which $A \subset A$?
- How are A and B related if $A \subseteq B$ and $B \subseteq A$?
- How are A and C related if $A \subset B$ and $B \subset C$?
- Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Power Set

- The power set of a set S, written $P(S)$, is the set of all subsets of S.
- Example: What is the powerset of $\{a, b, c\}$?
- Fact: If cardinality of S is n, then $|P(S)| = 2^n$
- What is the power set of $\emptyset$?
- What is the power set of $\{\emptyset\}$?

Ordered Tuples

- An important operation on sets is called Cartesian product
- But to define Cartesian produce, we first need to learn about discrete structure called ordered tuples
- Sets represent unordered collection of objects
- Ordered tuples represent ordered collections of objects
- An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection with a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its last element.

Sets vs. Ordered Tuples

- Consider sets $A = \{1, 2\}$ and $B = \{2, 1\}$. Is $A = B$?
- Consider ordered tuples $A = (1, 2)$ and $B = (2, 1)$. Is $A = B$?
- If a tuple has two elements, it's called a pair
- If tuple has three elements, it's called a triple
- When people say "tuple", this really means "ordered tuple"

Cartesian Product

- The Cartesian product of two sets A and B, written $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

- Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. What is $A \times B$?
- Example: What is $B \times A$?
- Observe: $A \times B \neq B \times A$ in general!

More on Cartesian Products

- If $|A| = n$ and $|B| = m$, what is $|A \times B|$?
- Cartesian product generalizes to more than two sets
- Cartesian product of $A_1 \times A_2 \ldots \times A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$
- Example: If $A = \{1, 2\}, B = \{a, b\}, C = \{\star, \circ\}$, what is $A \times B \times C$?

Set Operations

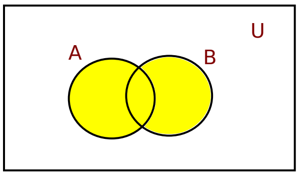
Four kinds of set operations:

- Union: Analogous to $\vee$ in boolean logic
- Intersection: Analogous to $\wedge$ in boolean logic
- Complement: Analogous to $\neg$ in boolean logic
- Difference: "Subtraction" of one set from another

Set Union

- The union of A and B, written $A \cup B$, is the set that contains those elements that are either in A or in B:

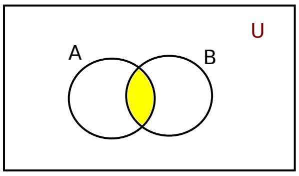
$$A \cup B = \{x \mid x \in A \ \vee x \in B\}$$



Set Intersection

- The intersection of A and B, written $A \cap B$, is the set that contains those elements that are both in A and in B:

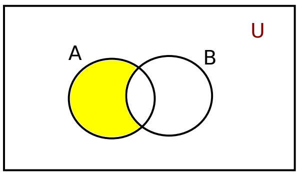
$$A \cap B = \{x \mid x \in A \ \wedge x \in B\}$$



Set Union

- The difference of A and B, written $A - B$, is the set that contains those elements that are in A but not in B:

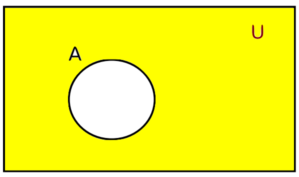
$$A - B = \{x \mid x \in A \ \wedge x \notin B\}$$



Set Complement

- The complement of a set A, written $\overline{A}$, is the set that contains only those elements that are not in A

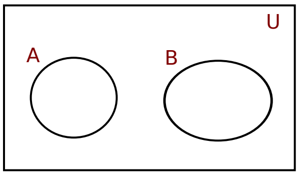
$$\overline{A} = \{x \mid x \notin A\}$$



- $\overline{A}$ is same as $U - A$

Disjoint Sets

- Two set A and B are called disjoint if $A \cap B = \emptyset$



Exercise

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- To prove equality between sets X and Y, we need to prove $X \subseteq Y$ and $Y \subseteq X$
- Similar to proving $P \Leftrightarrow Q$
- We'll first prove $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$
- Then prove $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$
- These two proofs establish the proof of $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof, Part I

- Let's first prove $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$
- We need to show if $x \in \overline{A \cup B}$, then $x \in \overline{A} \cap \overline{B}$
- If $x \in \overline{A \cup B}$, then by definition of complement, $\neg(x \in A \cup B)$
- Using definition of $\cup$, this implies $\neg(x \in A \ \vee \ x \in B)$
- Using DeMorgan's law, this is equivalent to $x \notin A \wedge x \notin B$
- Using definition of complement, $x \in \overline{A} \wedge x \in \overline{B}$
- Using definition of $\cap$, we have $x \in \overline{A} \cap \overline{B}$
- Thus, we've shown $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ $\square$

Proof, Part II

- Now, we need to prove $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$
- We need to show if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A \cup B}$
- If $x \in \overline{A} \cap \overline{B}$, then by definition of $\cap$, $x \in \overline{A} \wedge x \in \overline{B}$
- By definition of complement, $\neg(x \in A) \wedge \neg(x \in B)$
- Using DeMorgan's law, this is equivalent to $\neg(x \in A \vee x \in B)$
- Using definition of $\cup$, $\neg(x \in A \cup B)$
- Using definition of complement, we have $x \in \overline{A \cup B}$
- Thus, we've shown $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$ $\square$

Set Equivalences as Logical Equivalences

- We've just shown $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- Recall: $\cup$ behaves like $\vee$, $\cap$ behaves like $\wedge$, and complement behaves like $\neg$
- Interpreted this way, above formula corresponds to what logical equivalence?
- But this is exactly DeMorgan's law!
- The above equivalence also called DeMorgan's law for sets
- In general, logical equivalences translate to set equivalences
- See Rosen book for a comprehensive list of set equivalences

Proving Distributivity of $\cap$

- Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- As before, we need to prove both $\subseteq$ and $\supseteq$
- Proof of $\subseteq$: Need to show $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$

Proof, Part II

- Now, let's prove $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$

One More Example

- Prove A and $\overline{A}$ are disjoint
- Need to prove $A \cap \overline{A} = \emptyset$
- i.e., need to show there is no element x such that $x \in A \cap \overline{A}$
- Proof by contradiction: Suppose there is such an x
- By $\cap$ def, $x \in A \land x \in \overline{A}$
- By complement def, $x \in A \land \neg(x \in A)$
- But this a contradiction, thus $A \cap \overline{A} = \emptyset$ $\square$

Naive Set Theory and Russell's Paradox

- The intutive definition of sets we learned today goes back to German mathematician George Cantor (1800's)
- Cantor's set theory called naive because it can lead to paradoxes, which are logical inconsistencies
- In 1901, British mathematician Bertrand Russell showed that Cantor's set theory is called inconsistent
- This can be shown using so-called Russell's paradox

Russell's Paradox

- Let R be the set of sets that are not members of themselves:

$$R = \{S \mid S \notin S\}$$

- Two possibilities: Either $R \in R$ or $R \notin R$
- Suppose $R \in R$.
- But by definition of R, R does not have itself as a member, i.e., $R \notin R$
- But this contradicts $R \in R$
- Therefore, first possibility is infeasible

Russell's Paradox, cont.

- Now suppose $R \notin R$ (i.e., R not a member of itself)
- But since R is the set of sets that are not members of themselves, R must be a member of R
- But this implies $R \in R$, again yielding a contradiction
- Therefore, either possibility yields to a contradiction

Consequences of Russell's Paradox

- Russell's paradox shows that Cantor's formulation of set theory is inconsistent b/c it can yield paradoxes
- Inconsistent because possible to define sets that do not exist!
- Much research on consistent versions of set theory $\Rightarrow$ axiomatic set theories
- The version we learned in class is Cantor's original set theory because it is simple and intuitive, but realize that it can lead to logical inconsistencies...

Illustration of Russell's Paradox

- Russell's paradox and other similar paradoxes inspired artists at the turn of the century, esp. Escher and Magritte
- French painter Rene Magritte made a graphical illustration of Russel's paradox:

Escher's Illustration of Paradoxes

- Dutch painter Escher also inspired by mathematical paradoxes:

Paradoxes in Math, Logic, and CS

- Paradoxes are common in meta mathematics (study of math using mathematical methods):
  - Godel's incompleteness theorem: All consistent formulations of number theory include undecidable propositions
  - Turing's halting problem: Does there exist a program P' that can decide if any arbitrary program P terminates?
- More of these kinds of inconsistency results in this class and other CS classes
- If paradoxes interest you, read the book "Godel, Escher, Bach" by Douglas Hofstadter