# CS243: Discrete Structures

# Sequences, Summations, and Cardinality of Infinite Sets

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## Announcements

- Homework 2 is graded, scores on Blackboard
- Graded HW and sample solutions given at end of this lecture
  - Make sure score matches the one on Blackboard
  - If not, let us know within one week
- Mean on homework 2: $66/90$ (73%)
- Many of you made mistakes on questions 2, 3
- Similar questions will be on midterm – review the sample solutions!



## Midterm

- Midterm next Tuesday, Oct. 2, in lecture
- Midterm will cover all topics up to today's lecture:
  - propositional logic, first order logic, inference rules, proof techniques, sets, functions
- Good way to prepare is to go over lectures and homeworks
- To help you prepare, will give solutions for HW3 on Thursday
- Also, Thursday's lecture will be a review session
  - Weilin will go over difficult homework problems



## More on Midterm

- Midterm is closed-book, closed-notes, closed-phones, closed-laptops, closed-tablets etc.
- But you are allowed to bring three sheets of hand-written or typed notes **prepared by you**
- I'm out of town until next Wednesday, therefore Weilin will be proctoring the midterm



## Sequences

- A sequence is a discerete structure to represent an ordered list.
- Example: $1, 2, 3, 5, 8$ is a finite sequence with five terms
- Example: $1, 2, 3, 5, 8, 13, 21, \ldots$ is an infinite sequence (Fibonacci numbers)
- Formally, sequence is a function from a subset of $\mathbb{Z}$ to a set $S$.
- Notation $a_n$ represents $n$'th term in the sequence
- For Fibonacci sequence, $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 5$ etc.



## Sequence Examples

- What is the sequence defined by $a_n = \frac{1}{n}$ for $(n \geq 1)$?
- What is the sequence defined by $a_n = n^2$ for $n \geq 1$?
- What is the sequence defined by $a_n = (-1)^n$ for $n \geq 0$?

## Arithmetic Progression

- Some kinds of sequences come up a lot in discete math
- An arithmetic progression is a sequence of the form:

$$a, a+d, a+2d, a+3d, \ldots$$

- Here, the real number $a$ is called the initial term
- Also, $d$ is called the common difference
- Example: $2, 5, 8, 11, \ldots$
- What is the common difference?
- What is the initial term?



## Arithmetic Progression, cont.

- Arithmetic progressions can always be written as $a_n = a_0 + d \cdot n$ where $a_0$ is the initial term and $d$ is the common difference
- Example: $a_n = -1 + 4n$ for $n \geq 0$ is arithmetic progression with elements:

$$-1, 3, 7, 11, 15, \ldots$$

- Example: What is the closed-form definition for the sequence $2, 5, 8, 11, \ldots$?



## Geometric Progression

- Another class of sequences that come up often are geometric progressions
- A geometric progression is a sequence of the form:

$$a, ar, ar^2, ar^3, \ldots$$

- Here, $a$ is called the initial term, and $r$ is the common ratio
- Example: $1, -3, 9, -27, \ldots$
- What is the initial term?
- What is the common ratio?



## Geometric Progression, cont.

- Geometric progressions can always be written as $a_n = a_0 \cdot r^n$ where $a_0$ is the initial term and $r$ is the common ratio
- Example: The sequence defined by $a_n = 6 \cdot (\frac{1}{3})^n$ for $n \geq 0$ is an geometric progression with elements:

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27} \ldots$$

- Example: What is the closed-form definition for the sequence $1, -3, 9, -27, \ldots$?



## Summations

- Given a sequence, one common operation is to sum up all the terms in that sequence
- For this purpose, we use the summation notation:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

- Example: $$\sum_{j=1}^{3} j \; = \; 1 + 2 + 3 = 6$$
- The variable $j$ in this notation is called index of summation
- Also, $m$ and $n$ are called the lower and upper limits of the summation



## Summation Examples

- Consider the sequence $a_n = n^2$. What is the value of this summation?

$$\sum_{j=1}^{4} a_j$$

- What is the value of this summation?

$$\sum_{j=4}^{8} (-1)^j$$

## Nested Summations

- It is also common to nest summations within one another.
- What is the value of the following summation?

$$\sum_{i=1}^{2}\sum_{j=1}^{3} ij$$

- Answer:

$$\sum_{i=1}^{2}\sum_{j=1}^{3} ij = \sum_{i=1}^{2} i + 2i + 3i = \sum_{i=1}^{2} 6i = 6 + 12 = 18$$



## Closed Form of Summations

- Some summations arise all the time in discrete mathematics
- Example: Sum of all numbers from 1 to $n$: $\sum_{i=1}^{n} i$
- For such common summations, it is often useful to derive a closed form
- The closed form expresses the value of the summation as a formula without summations
- The closed form of above summation is:

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$



## Example

- Compute the value of the summation:

$$\sum_{i=21}^{50} i = 1275 - 210 = 1065$$

- We can rewrite this summation as:

$$\sum_{i=1}^{50} i - \sum_{i=1}^{20} i$$

- By previous definition, first summation is:
- Second summation is:



## Useful Property Summations

- Given a summation consisting of addition and multiplication terms, we can decompose it as follows:

$$\sum_{j=m}^{n} (ax_j + by_j) = a\sum_{j=m}^{n} x_j + b\sum_{j=m}^{n} y_j$$

- Example: Compute the value of $\sum_{i=1}^{10} 3i + 2$
- This can be written as $3\sum_{i=1}^{10} i + \sum_{i=1}^{10} 2$



## Arithmetic Series

- The sum of the terms in an airthmetic progression $a, a + d, a + 2d, \ldots$ is called an arithmetic series.
- Let's derive a closed form for arithmetic series:

$$\sum_{i=1}^{n} (a + di)$$

- By the earlier property, we can write this as:

$$\sum_{i=1}^{n} a + d\sum_{i=1}^{n} i$$



## Closed Form for Arithmetic Series

$$\sum_{i=1}^{n} (a + di) = \sum_{i=1}^{n} a + d\sum_{i=1}^{n} i$$

- What is $\sum_{i=1}^{n} a$?
- By earlier closed form, we have:

$$\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

- Thus, we can write entire arithmetic series in closed form as:

$$an + \frac{dn(n+1)}{2}$$

## Example

- What is the closed form for the following summation?

$$\sum_{j=3}^{n}(2+3j)$$

- Trick: Write this as the difference of two summations:

$$\sum_{j=1}^{n}(2+3j)-\sum_{j=1}^{2}(2+3j)$$

- Expand the second term:

$$\sum_{j=1}^{n}(2+3j)-13$$



## Example, cont.

$$\sum_{j=1}^{n}(2+3j)-13$$

- Now, compute the closed form for first term:

$$\sum_{j=1}^{n}2+3\sum_{j=1}^{n}j-13$$

- Using known closed forms, this can be rewritten as:

$$2n+3\frac{n(n+1)}{2}-13$$



## Geometric Series

- The sum of the terms in a geometric progression $a, ar, ar^2, \dots$ is called a geometric series
- Theorem: Closed form of geometric series ($r \neq 1$):

$$\sum_{j=0}^{n}(ar^j)=a\cdot\frac{r^{n+1}-1}{r-1}$$

- This is very useful to know– memorize it!
- Let's prove why this closed form is correct



## Derivation of Geometric Series Closed Form

Theorem: Closed form of geometric series ($r \neq 1$):

$$\sum_{j=0}^{n}(ar^j)=a\cdot\frac{r^{n+1}-1}{r-1}$$

- First, let's call the summation on left $S$
- Now, let's multiply $S$ by r:

$$rS=r\sum_{j=0}^{n}ar^j=\sum_{j=0}^{n}ar^{j+1}$$



## Derivation continued

$$rS=\sum_{j=0}^{n}ar^{j+1}$$

- Now, change index of summation from $j$ to $k$ where $k=j+1$:

$$rS=\sum_{j=0}^{n}ar^{j+1}=\sum_{k=1}^{n+1}ar^k$$

- Now, rewrite this as:

$$rS=\sum_{k=1}^{n+1}ar^k=\sum_{k=0}^{n}ar^k-a+ar^{n+1}$$



## Derivation, cont.

$$rS=\sum_{k=0}^{n}ar^k-a+ar^{n+1}$$

- Now, observe first term on left hand side is $S$!
- Thus, we have:

$$rS=S+ar^{n+1}-a$$

- Collecting $S$ on one side, we get:

$$S=a\frac{r^{n+1}-1}{r-1}$$

- This is exactly the closed form from the theorem!

## Example 1

$$\sum_{j=0}^{n}(ar^j) = a \cdot \frac{r^{n+1}-1}{r-1}$$

- Compute the value of $\sum_{i=0}^{5} 3 \cdot 2^i$
- What is $a$?
- What is $r$?
- Using closed form, we have:

$$\sum_{i=0}^{5} 3 \cdot 2^i = 3 \cdot \frac{2^6-1}{2-1} = 189$$



## Example 2

- For $|r| < 1$, derive a closed form for the summation

$$\sum_{n=0}^{\infty} a \cdot r^n$$

- Using closed form for geometric series, this is equivalent to:

$$\lim_{n \to \infty} a \cdot \frac{r^{n+1}-1}{r-1}$$

- Since $|r| < 1$, $r^{n+1}$ becomes $0$ as $n$ approaches infinity
- Thus, this is equivalent to:

$$a \cdot \frac{0-1}{r-1} = \frac{a}{1-r}$$



## Example 3

- Compute the value of the summation:

$$\sum_{k=0}^{\infty} 3 \cdot (\frac{1}{9})^k = \frac{3}{1-\frac{1}{9}} = \frac{27}{8}$$

- Using previous formula, this sum is given by $\frac{a}{1-r}$



## Revisiting Sets

- Earlier we talked about sets and cardinality of sets
- Recall: Cardinality of a set is number of elements in that set
- This definition makes sense for sets with finitely many element, but more involved for infinite sets
- Agenda: Revisit the notion of cardinality for infinite sets



## Cardinality of Infinite Sets

- Sets with infinite cardinality are classified into two classes:
  1. Countably infinite sets (e.g., natural numbers)
  2. Uncountably infinite sets (e.g., real numbers)
- A set $A$ is called countably infinite if there is a bijection between $A$ and the set of positive integers.
- A set $A$ is called countable if it is either finite or countably infinite
- Otherwise, the set is called uncountable or uncountably infinite



## Example

Prove: The set of odd positive integers is countably infinite.

- Need to find a function $f$ from $\mathbb{Z}^+$ to the set of odd positive integers, and prove that $f$ is bijective
- Consider $f(n) = 2n - 1$ from $\mathbb{Z}^+$ to odd positive integers
- We need to show $f$ is bijective (i.e., one-to-one and onto)
- Let's first prove injectivity, then surjectivity

## Example, cont.

Prove injectivity of $f(n) = 2n - 1$

- Recall: Function is injective if $f(a) = f(b) \rightarrow a = b$
- Suppose $f(a) = f(b)$. Then $2a - 1 = 2b - 1$
- This implies $a = b$, establishing injectivity



## Example, cont.

Prove surjectivity of $f(n) = 2n - 1$

- Recall: Function is surjective if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$
- Proof by contradiction: Suppose there is some odd positive integer $b$ such that $\forall x \in \mathbb{Z}^+ . 2x - 1 \neq b$
- This implies $\frac{b+1}{2}$ is not an integer.
- But since $b$ is an odd positive integer, $b + 1$ is even
- Thus, $b + 1$ is divisible by $2$, yielding a contradiction.
- Since we showed that there is a bijection (namely $2n - 1$) from positive integers to odd positive integers, the set of odd positive integers is countably infinite



## Another Way to Prove Countable-ness

- One way to show a set $A$ is countably infinite is to give bijection between $\mathbb{Z}^+$ and $A$
- Another way is by showing members of $A$ can be written as a sequence $(a_1, a_2, a_3, \ldots)$
- Since such a sequence is a bijective function from $\mathbb{Z}^+$ to $A$, writing $A$ as a sequence $a_1, a_2, a_3, \ldots$ establishes one-to-one correspondence



## Another Example

Prove that the set of all integers is countable

- We can list all integers in a sequence, alternating positive and negative integers:

$$a_n = 0, 1, -1, 2, -2, 3, -3, \ldots$$

- Observe that this sequence defines the bijective function:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n-1)/2 & \text{if } n \text{ odd} \end{cases}$$



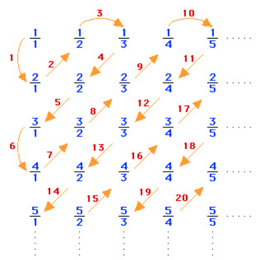
## Rational Numbers are Countable

- Not too surprising $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite
- More surprising: Set of rationals is also countably infinite!
- We'll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence
- Recall: Every positive rational number can be written as the quotient $p/q$ of two positive integers $p, q$

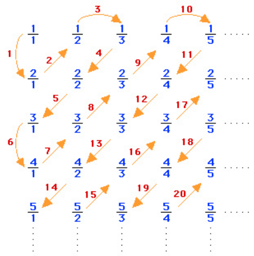


## Rationals in a Table

- Now imagine placing rationals in a table such that:
  1. Rationals with $p = 1$ go in first row, $p = 2$ in second row, etc.
  2. Rationals with $q = 1$ in 1st column, $q = 2$ in 2nd column, ...



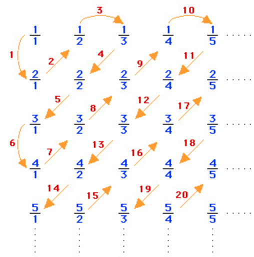
## Enumerating the Rationals




- How to enumerate entries in this table without missing any?
- Trick: First list those with $p + q = 2$, then $p + q = 3$, ...
- Traverse table diagonally from left-to-right, in the order shown by arrows



## Enumerating the Rationals, cont.




- This allows us to list all rationals in a sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \ldots$$

- Hence, set of rationals is countable



## Uncountability of Real Numbers

- Prime example of uncountably infinite sets is real numbers
- The fact that $\mathbb{R}$ is uncountably infinite was proven by George Cantor using the famous Cantor's diagonalization argument
- This was a shocking result in mathematics in the 1800's
- This argument has inspired many similar famous proofs in the theory of computation
- Brief look at the diagonalization argument



## Cantor's Diagonalization Argument

- For contradiction, assume the set of reals was countable
- Since any subset of a countable set is also countable, this would imply the set of reals between $0$ and $1$ is also countable
- Now, if reals between $0$ and $1$ are countable, we can list them in a table in some order:

$$\begin{array}{lllllll}
R_1 = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\
R_2 = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\
R_3 = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\
\vdots & & & & & \ddots & \vdots & \\
R_n = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}$$



## Diagonalization Argument, cont

$$\begin{array}{lllllll}
R_1 = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\
R_2 = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\
R_3 = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\
\vdots & & & & & \ddots & \vdots & \\
R_n = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}$$

- Now, create a new real number $R = 0.a_1 a_2 a_3 \ldots$ such that:

$$a_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$$

- Clearly, this new number $R$ differs from each number $R_i$ in the table in at least one digit (its $i$'th digit)



## Diagonalization Argument, concluded

- Since $R$ is not in the table, this is not a complete enumeration of all reals between $0$ and $1$
- Hence, the set of real between $0$ and $1$ is not countable
- Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite

□