CS243: Discrete Structures

Introduction to Number Theory

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Announcements

- Midterms are graded – scores on Blackboard
  - Graded midterms handed back at the end of class
  - Make sure score on midterm matches grade on Blackboard
  - If not, let us know asap (within a week at latest)
- Sample solutions posted on course webpage – look over them!
- Since only 1-2 students answered Question (4c) correctly and since there was small typo, did not count (4c) when calculating grades – thus, grades out of 80 rather than 90

Announcements, cont.

- Average on midterm 55 out of 80; standard deviation is 17
- Third homework also graded – scores on Blackboard
- Average on HW3 is 67 out of 100

Introduction

- Number theory is the branch of mathematics that deals with integers and their properties
- Number theory has a number of applications in computer science, esp. in modern cryptography
- This lecture: Important results in number theory
- Next lecture: Continue discussion of number theory, look at applications of number theory in cryptography

Divisibility

- Given two integers a and b where $a \neq 0$, we say a divides b if there is an integer c such that $b = ac$
- If a divides b, we write $a|b$; otherwise, $a \not| \, b$
- Example: $2|6$, $2 \not| \, 9$
- If $a|b$, a is called a factor of b
- b is called a multiple of a
- All integers divisible by a can be enumerated as:

$$\ldots, -3a, -2a, -a, 0, a, 2a, 3a, \ldots$$

Example

- Question: If n and d are positive integers, how many integers not exceeding n are divisible by d?
- Recall: All positive integers divisible by d are of the form dk
- We want to find how many numbers dk there are such that $0 < dk \leq n$.
- In other words, we want to know how many integers k there are such that $0 < k \leq \frac{n}{d}$
- How many integers are there between 1 and $\frac{n}{d}$?

Properties of Divisibility

- Theorem 1: If $a|b$ and $a|c$, then $a|(b+c)$
- Proof: If $a|b$, then $\exists\ k_1$ such that $b = ak_1$
- Similarly, if $a|b$, then $\exists\ k_2$ such that $c = ak_2$
- Then, $b + c = a(k_1 + k_2)$
- Hence, $a|(b+c)$ □

More Divisibility Properties

- Theorem 2: If $a|b$, then $a|bc$ for all integers c
- Proof: If $a|b$, then there exists k such that $b = ak$.
- Hence, $bc = a \cdot ck$
- Therefore, $a|bc$ □

Divisibility Properties, cont.

- Theorem 3: If $a|b$ and $b|c$, then $a|c$
- Proof: If $a|b$, there exists k_1 such that $b = ak_1$
- Since $b|c$, there exists k_2 such that and $c = bk_2$.
- This implies $c = a \cdot k_1 k_2$.
- Hence $a|c$. □

Divisibility Properties, cont.

- Theorem 4 If $a|b$ and $a|c$, then $a|(mb + nc)$
- Proof: By Thm 2, if $a|b$, then $a|mb$
- By thm 2, if $a|c$, then $a|nc$.
- By Thm 1, if $a|mb$ and $a|nc$, then $a|(mb + nc)$

The Division Theorem

- Division theorem: Let a be an integer, and d a positive integer. Then, there are unique integers q, r with $0 \leq r < d$ such that $a = dq + r$
- Here, d is called divisor, and a is called dividend
- q is the quotient, and r is the remainder.
- We use the $r = a \bmod d$ notation to express the remainder
- The notation $q = a \text{ div } d$ expresses the quotient
- What is $101 \bmod 11$?
- What is $101 \text{ div } 11$?

Congruence Modulo

- Sometimes, we care if two integers a, b have the same remainder when divided by some number m.
- If so, a and b are congruent modulo m, $a \equiv b \pmod{m}$.
- More technically, if a and b are integers and m a positive integer, $a \equiv b \pmod{m}$ iff $m|(a-b)$
- Example: 7 and 13 are congruent modulo 3.
- Example: Find a number congruent to 7 modulo 4.

Congruence Modulo Theorem

- Theorem: $a \equiv b \pmod{m}$ iff $a \bmod m = b \bmod m$
- Part 1, $\Rightarrow$: Suppose $a \equiv b \pmod{m}$.
- Then, by definition of $\equiv$, $m|(a-b)$
- By definition of $|$, there exists k such that $a - b = mk$, i.e., $a = b + mk$
- By division thm, $b = mp + r$ for some $0 \leq r < m$
- Then, $a = mp + r + mk = m(p+k) + r$
- Thus, $a \bmod m = r = b \bmod m$

Congruence Modulo Theorem Proof, cont.

- Theorem: $a \equiv b \pmod{m}$ iff $a \bmod m = b \bmod m$
- Part 2, $\Leftarrow$: Suppose $a \bmod m = b \bmod m$
- Then, there exists some p_1, p_2, r such that $a = p_1 \cdot m + r$ and $b = p_2 \cdot m + r$ where $0 \leq r < m$
- Then, $a - b = p_1 \cdot m + r - p_2 \cdot m - r = m \cdot (p_1 - p_2)$
- Thus, $m|(a-b)$
- By definition of $\equiv$, $a \equiv b \pmod{m}$

Example

- Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:

$$a + c \equiv b + d \pmod{m}$$

- Proof: Since $a \equiv b \pmod{m}$, $m|(a-b)$
- Since $c \equiv d \pmod{m}$, $m|(c-d)$
- By definition of $|$, there exists k_1, k_2 such that:

$$a - b = mk_1 \quad c - d = mk_2$$

- Adding these, we get: $a + c - (b + d) = m(k_1 + k_2)$
- Again, by definition of $\equiv$ and $|$, this means $a + c \equiv b + d \pmod{m}$ $\square$

Applications of Congruence in Cryptography

- Congruences have many applications in cryptography
- For instance, Julius Caesar encrypted messages by shifting each letter three letters in the alphabet ("Caesar cipher")
- For example, the message "I LIKE DISCRETE MATH" would be encrypted as "L OLNH GLYFUHVH PDVK"
- Caesar's cipher example of shift cipher: shifts each letter by k
- For Caesar cipher, $k = 3$
- We can express express shift ciphers using the modulo operator

Shift Ciphers

- First, let's number letters A-Z with $0 - 25$
- Represent message with sequence of numbers
- Example: The sequence "25 0 2" represents "ZAC"
- To encrypt, apply encryption function f defined as:

$$f(x) = (x + k) \bmod 26$$

- Because f is bijective, its inverse yields decryption function:

$$f^{-1}(x) = (x - k) \bmod 26$$

Ciphers and Congruence Modulo

- Shift cipher is a very primitive and insecure cipher because very easy to infer what k is
- But contains some useful ideas:
  - Encoding words as sequence of numbers
  - Use of modulo operator
- Modern encryption schemes much more sophisticated, but also share these principles
- More on this next lecture!

Prime Numbers

- A positive integer p that is greater than 1 and divisible only by 1 and itself is called a prime number.
- First few primes: $2, 3, 5, 7, 11, \ldots$
- A positive integer that is greater than 1 and that is not prime is called a composite number
- Example: $2, 4, 6, 8, 9, \ldots$

Fundamental Theorem of Arithmetic

- Fundamental Thm: Every positive integer greater than 1 is either prime or can be written uniquely as a product of primes.
- This unique product of prime numbers for x is called the prime factorization of x
- Examples:
  - $12 =$
  - $21 =$
  - $99 =$

Determining Prime-ness

- In many applications, such as crypto, important to determine if a number is prime – following thm is useful for this:
- Theorem: If n is composite, then it has a prime divisor less than or equal to $\sqrt{n}$
- Proof: Since n is composite, it can be written as $n = ab$ where $a > 1$ and $b > 1$.
- For contradiction, suppose neither a nor b are $\leq \sqrt{n}$, i.e., $a > \sqrt{n}$, $b > \sqrt{n}$
- Then, $n = ab > \sqrt{n}^2 = n$, a contradiction.
- Hence, either $a \leq \sqrt{n}$, or $b \leq \sqrt{n}$, and by the Fundamental Thm, is either itself a prime or has a factor less than itself. $\square$

Consequence of This Theorem

Theorem: If n is composite, then it has a prime divisor $\leq \sqrt{n}$

- Thus, to determine if n is prime, only need to check if it is divisible by primes $\leq \sqrt{n}$
- Example: Show that 101 is prime
- Since $\sqrt{101} < 11$, only need to check if it is divisible by $2, 3, 5, 7$.
- Since it is not divisible by any of these, we know it is prime.

Infinitely Many Primes

- Theorem: There are infinitely many prime numbers.
- Proof: (by contradiction) Suppose there are finitely many primes: $p_1, p_2, \ldots, p_n$
- Now consider the number $Q = p_1 p_2 \ldots p_n + 1$. Q is either prime or composite
- Case 1: Q is prime. We get a contradiction, because we assumed only prime numbers are $p_1, \ldots, p_n$
- Case 2: Q is composite. In this case, Q can be written as product of primes.
- But Q is not divisible by any of $p_1, p_2, \ldots, p_n$
- Hence, by Fundamental Thm, not composite $\Rightarrow \bot$ $\square$

A Word about Prime Numbers and Cryptography

- Prime numbers play a key role in modern cryptography
- Modern cryotpgraphy techniques rely on prime numbers to encrypt messages
- Security of encryption relies on prime factorization being intractable for sufficiently large numbers
- More on this later...

Greatest Common Divisors

- Suppose a and b are integers, not both 0.
- Then, the largest integer d such that $d|a$ and $d|b$ is called greatest common divisor of a and b, written gcd(a,b).
- Example: $\gcd(24, 36) =$
- Example: $\gcd(2^3 5, 2^2 3) =$
- Example: $\gcd(14, 25) =$
- Two numbers whose gcd is 1 are called relatively prime.
- Example: 14 and 25 are relatively prime

Least Common Multiple

- The least common multiple of a and b, written lcm(a,b), is the smallest integer c such that $a|c$ and $b|c$.
- Example: lcm$(9, 12)$=
- Example: lcm$(2^3 3^5 7^2, 2^4 3^3)$=

Theorem about LCM and GCD

- Theorem: Let a and b be positive integers. Then, $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$
- Proof: Let $a = p_1^{i_1} p_2^{i_2} \dots p_n^{i_n}$ and $b = p_1^{j_1} p_2^{j_2} \dots p_n^{j_n}$
- Then, $ab = p_1^{i_1+j_1} p_2^{i_2+j_2} \dots p_n^{i_n+j_n}$
- $\gcd(a, b) = p_1^{min(i_1,j_1)} p_2^{min(i_2,j_2)} \dots p_n^{min(i_n,j_n)}$
- $\text{lcm}(a, b) = p_1^{max(i_1,j_1)} p_2^{max(i_2,j_2)} \dots p_n^{max(i_n,j_n)}$
- Thus, we need to show $i_k + j_k = min(i_k, j_k) + max(i_k, j_k)$

Proof, cont.

- Show $i_k + j_k = min(i_k, j_k) + max(i_k, j_k)$
- Either (i) $i_k < j_k$ or (ii) $i_k \geq j_k$
- If (i), then $min(i_k, j_k) = i_k$ and $max(i_k, j_k) = j_k$
- Thus, $i_k + j_k = min(i_k, j_k) + max(i_k, j_k)$
- If (ii), then $min(i_k, j_k) = j_k$ and $max(i_k, j_k) = i_k$
- Hence $min(i_k, j_k) + max(i_k, j_k) = i_k + j_k$ $\square$

Computing GCDs

- Simple algorithm to compute gcd of a, b:
  - Factorize a as $p_1^{i_1} p_2^{i_2} \dots p_n^{i_n}$
  - Factorize b as $p_1^{j_1} p_2^{j_2} \dots p_n^{j_n}$
  - $\gcd(a, b) = p_1^{min(i_1,j_1)} p_2^{min(i_2,j_2)} \dots p_n^{min(i_n,j_n)}$
- But this algorithm is not very practical because prime factorization is computationally expensive!
- Much more efficient algorithm to compute gcd, called the Euclidian algorithm

Insight Behind Euclid's Algorithm

- Theorem: Let $a = bq + r$. Then, $\gcd(a, b) = \gcd(b, r)$
- Proof: We'll show that a, b and b, r have the same common divisors – implies they have the same gcd.

$\Rightarrow$ Suppose d is a common divisor of a, b, i.e., $d|a$ and $d|b$

- By theorem we proved earlier, this implies $d|a - bq$
- Since $a - bq = r$, $d|r$. Hence d is common divisor of b, r.

$\Leftarrow$ Now, suppose $d|b$ and $d|r$. Then, $d|bq + r$

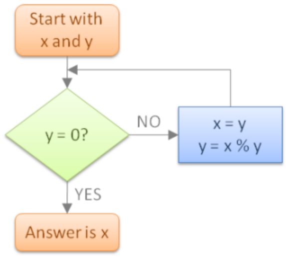
- Hence, $d|a$ and d is common divisor of a, b $\square$

Using this Theorem

Theorem: Let $a = bq + r$. Then, $\gcd(a, b) = \gcd(b, r)$

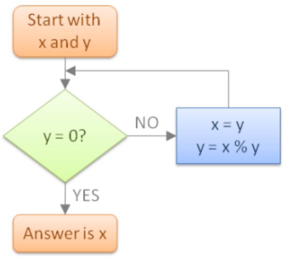
- Theorem suggests following strategy to compute gcd(a, b):
- Compute $r_1 = a \bmod b$ ($\gcd(a, b) = \gcd(b, r_1)$)
- Compute $r_2 = b \bmod r_1$ ($\gcd(a, b) = \gcd(r_1, r_2)$)
- Compute $r_3 = r_1 \bmod r_2$ ($\gcd(a, b) = \gcd(r_2, r_3)$)
- Repeat until remainder becomes 0 ($\gcd(a, b) = \gcd(r_n, 0) = r_n$)
- The last non-zero remainder is the gcd of a and b!

Euclidian Algorithm



- Find gcd of 72 and 20
- $12 = 72\%20$
- $8 = 20\%12$
- $4 = 12\%8$
- $0 = 8\%4$
- gcd is 4!

Euclidian Algorithm Example



- Find gcd of 662 and 414
- $248 = 662\%414$
- $166 = 414\%248$
- $82 = 248\%166$
- $2 = 166\%82$
- $0 = 82\%2$
- gcd is 2!

GCD as Linear Combination

- gcd(a, b) can be expressed as a linear combination of a and b
- Theorem: If a and b are positive integers, then there exist integers s and t such that:

$$\gcd(a, b) = s \cdot a + t \cdot b$$

- Furthermore, Euclidian algorithm gives us a way to compute these integers s and t

Example

- Express $\gcd(252, 198)$ as a linear combination of 252 and 198
- First apply Euclid's algorithm (write $a = bq + r$ at each step):
  1. $252 = 1 \cdot 198 + 54$
  2. $198 = 3 \cdot 54 + 36$
  3. $54 = 1 \cdot 36 + 18$
  4. $36 = 2 \cdot 18 + 0 \Rightarrow$ gcd is 18
- Now, using (3), write 18 as $54 - 1 \cdot 36$
- Using (2), write 18 as $54 - 1 \cdot (198 - 3 \cdot 54)$
- Using (1), we have $54 = 252 - 1 \cdot 198$, thus:

$$18 = (252 - 1 \cdot 198) - 1(198 - 3 \cdot (252 - 1 \cdot 198))$$

Example, cont.

$$18 = (252 - 1 \cdot 198) - 1(198 - 3 \cdot (252 - 1 \cdot 198))$$

- Now, let's simplify this:

$$18 = 252 - 1 \cdot 198 - 1 \cdot 198 + 3 \cdot 252 - 3 \cdot 198$$

- Now, collect all 252 and 198 terms together:

$$18 = 4 \cdot 252 - 5 \cdot 198$$

- Trace steps of Euclid's algorithm backwards to derive s, t:

$$\gcd(a, b) = s \cdot a + t \cdot b$$

- This is known as the extended Euclidian algorithm

A Useful Result

- Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.
- Proof: Since a, b are relatively prime $\gcd(a, b) = 1$
- By previous theorem, there exists s, t such that $1 = s \cdot a + t \cdot b$
- Multiply both sides by c: $c = csa + ctb$
- By earlier theorem, since $a|bc$, $a|ctb$
- Also, by earlier theorem, $a|csa$
- Therefore, $a|csa + ctb$, which implies $a|c$ since $c = csa + ctb$ □

Example

Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.

- Suppose $15 \mid 16 \cdot x$
- Here 15 and 16 are relatively prime
- Thus, previous theorem implies: $15|x$

Question

- Suppose $ca \equiv cb \pmod{m}$. Does this imply $a \equiv b \pmod{m}$?
- Counterexample: Consider $14 \equiv 8 \pmod{6}$
- Thus, $2 \cdot 7 \equiv 2 \cdot 4 \pmod{6}$
- But $7 \not\equiv 4 \pmod{6}$
- Therefore, this implication does not hold in the general case!
- However, if c and m are relatively prime, it does hold

Another Useful Result

- Theorem: If $ca \equiv cb \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$
- Proof: Since $ca \equiv cb \pmod{m}$, we have $m \mid ca - cb$
- Rewriting, we get: $m \mid c(a - b)$
- Since m, c are relatively prime, previous thm implies $m \mid a - b$
- By definition of congruence, $a \equiv b \pmod{m}$

Examples

- If $15x \equiv 15y \pmod{4}$, is $x \equiv y \pmod{4}$?
- If $8x \equiv 8y \pmod{4}$, is $x \equiv y \pmod{4}$?
- Counterexample: $8 \cdot 2 \equiv 8 \cdot 3 \pmod{4}$, but $2 \not\equiv 3 \pmod{4}$