CS243 Midterm 1
Fall 2012-13

- Please read all instructions (including these) carefully.
- There are 5 questions on the exam, all with multiple parts. You have 75 minutes to work on the exam.
- The exam is closed book, closed notes, closed computers, phones, tablets, etc. However, you may use 3 sheets of hand-written (or typed) notes prepared by you.
- Please write your answers in the space provided on the exam, and clearly mark your solutions. Please do not use any additional scratch paper.
- Solutions will be graded on correctness and clarity. Each problem has a relatively simple and straightforward solution. You may get as few as 0 points for a question if your solution is far more complicated than necessary. Partial solutions will be graded for partial credit.

NAME: ____

W&M email address: __________________________________

Student ID: ____

In accordance with the letter and the spirit of the W&M honor code I have neither given nor received assistance on this examination.

SIGNATURE: ____

Problem	Max points	Points
1	10	
2	25	
3	15	
4	25	
5	15	
Total	90	

1. Consider the following truth table for a propositional logic formula ϕ containing two variables p, q:

p	q	ϕ
T	T	F
T	F	T
F	T	T
F	F	T

   (a) (7 points) Give a propositional formula that is equivalent to ϕ. Your answer may only contain the propositional variables p, q and at most 3 boolean connectives. You should not use any connectives other than $\neg, \wedge, \vee, \rightarrow$.

   Solution: Possible answers include $\neg(p \wedge q)$, $\neg p \vee \neg q$, $p \rightarrow \neg q$

   (b) (3 points) State whether this formula is valid, unsatisfiable, or contingent. Justify your answer.

   Solution: This formula is contingent because it is satisfiable, but not valid. It is satisfiable because it has satisfying interpretations, e.g., $p =$T, $q =$F. However, it is not valid because it has a falsifying interpretation $p =$T, $q =$T.

2. (a) (6 points) Consider the universe of discourse $U = \{a, b\}$ and a unary predicate p with interpretation $p(a) = \text{false},\ p(b) = \text{true}$ and a binary predicate q with interpretation:

$$q(a, a) = \text{false},\ q(a, b) = \text{true},\ q(b, a) = \text{true},\ q(b, b) = \text{false}$$

State the truth value of the formula

$$\forall x, y.\ (q(x, y) \lor p(x) \to q(y, x))$$

under this universe of discourse and this interpretation. Justify your answer.

Solution: Under this domain and interpretation, the truth value of the formula is false because there are objects x, y for which $(q(x, y) \lor p(y)) \to q(y, x)$ is false. In particular, consider $x = b, y = b$. In this case, $q(b, b) = \text{false}$, $p(b) = \text{true}$. Therefore, the antecedent is true, but the right hand side of the implication is false.

Common Mistakes: Some students said the truth value of this formula is "contingent" or "valid". Observe that under a particular domain and a particular interpretation, the truth value of the formula is either true or false. Contingent means the formula evaluates to true in some domains and some interpretations and evaluates to false in others. Validity means the formula evaluates to true for all domains and all interpretations. Therefore, the concepts "satisfiability", "contingency", "validity" etc. do not make sense when we are talking one particular domain and one particular interpretation.

(b) (12 points) Consider the following first-order formula:

$$\forall x.\exists y.P(x,y) \rightarrow \exists x.\forall y.P(x,y)$$

State whether this formula is valid, contingent, or unsatisfiable. Prove your answer.

Solution: This formula is contingent. To prove this, we give one satisfying and one falsifying interpretation:

- Satisfying interpretation: $U = \{\star\}, P(\star,\star) = \text{true}$. In this case, both the left-and-side and the right-hand-side of the implication is true, hence the formula evaluates to true.
- Falsifying interpretation: $U = \{\star, \circ\}, P(\star,\star) = \text{true}, P(\star,\circ) = \text{false}, P(\circ,\star) = \text{false}, P(\star,\star) = \text{true}$. Under this interpretation, the left hand side of the implication is true because for all x (namely, $\star$ and $\circ$), there exists a y (namely x itself) for which $p(x,y)$ is true. However, the righthand side of the implication is false because both $P(\star,\circ)$ and $P(\circ,\star)$ are false.

Common Mistakes: Many students made mistakes on this question. A common mistake was to argue that the formula is valid, contingent, or unsatisfiable for a particular domain and interpretation. As noted under "Common Mistakes" for part (a), a formula is either true or false under a particular domain and interpretation. To prove a formula is contingent, you must provide (a) a domain and interpretation under which the formula evaluates to true, and (b) a domain and interpretation under which the formula evaluates to false. Under the same interpretation and domain, the same formula cannot evaluate to both true and false.

(c) (7 points) Prove that the formulas $\neg(\forall x.(p(x) \rightarrow (q(x) \rightarrow \neg r(x))))$ and $\exists x.\ (p(x) \wedge q(x) \wedge r(x))$ are equivalent. Clearly label any equivalences you use.

Solution: These formulas are equivalent because we can apply known logical equivalences to rewrite the first formula into the second formula.

1.	$\neg(\forall x.(p(x) \rightarrow (q(x) \rightarrow \neg r(x))))$	
2.	$\neg(\forall x.(p(x) \rightarrow (\neg q(x) \vee \neg r(x))))$	remove $\rightarrow$
3.	$\neg(\forall x.(\neg p(x) \vee (\neg q(x) \vee \neg r(x))))$	remove $\rightarrow$
4.	$\neg(\forall x.(\neg p(x) \vee \neg q(x) \vee \neg r(x)))$	associativity
5.	$(\exists x.\neg(\neg p(x) \vee \neg q(x) \vee \neg r(x)))$	DeMorgan
6.	$\exists x.(\neg\neg p(x) \wedge \neg\neg q(x) \wedge \neg\neg r(x))$	DeMorgan
7.	$\exists x.(p(x) \wedge q(x) \wedge r(x))$	Double negation

3. (15 points) Give a formal proof using logical inference rules that the conclusion $\exists x. A(x) \wedge B(x) \wedge C(x)$ follows from the premises below. Your proof must clearly label which inference rules are used and how they are used.

(1)	$\forall x.\ A(x) \to C(x)$	Premise
(2)	$\forall x.\ \neg B(x) \to \neg C(x)$	Premise
(3)	$\exists x. A(x)$	Premise

Solution:

(1)	$\forall x.\ A(x) \to C(x)$	Premise
(2)	$\forall x.\ \neg B(x) \to \neg C(x)$	Premise
(3)	$\exists x. A(x)$	Premise
(4)	$A(a)$	$\exists$ instantiation (3)
(5)	$A(a) \to C(a)$	$\forall$ instantiation (1)
(6)	$C(a)$	Modus ponens (4), (5)
(7)	$\neg B(a) \to \neg C(a)$	$\forall$ instantiation (2)
(8)	$B(a)$	Modus tollens (6), (7)
(9)	$A(a) \wedge B(a)$	$\wedge$ intro (4), (8)
(10)	$A(a) \wedge B(a) \wedge C(a)$	$\wedge$ intro (9), (6)
(11)	$\exists x.\ (A(x) \wedge B(x) \wedge C(x))$	$\exists$ intro(10)

4. (a) (10 points) Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x+1}$. Prove that f is one-to-one. Your answer should clearly state any proof strategies used in the proof.

Solution:

A function is one-to-one if $\forall x, y.\ f(x) = f(y) \to x = y$. To prove this, we will use a direct proof strategy. That is, we will assume that $f(x) = f(y)$ and show that this implies $x = y$. Since $f(x) = \frac{1}{x+1}$ and $f(y) = \frac{1}{y+1}$ and since we assume $f(x) = f(y)$:

$$\frac{1}{x+1} = \frac{1}{y+1}$$

Rewriting this, we obtain:

$$y + 1 = x + 1$$

which implies $x = y$. Therefore, f is one-to-one.

(b) (5 points) Is the function f defined as in part (a) a bijection? If yes, prove your answer; if no, provide a counterexample.

Solution:

f is not a bijection because it is not onto. In particular, there are real numbers which are not the image of any integer under f. For example, there is no integer x such that $f(x) = 2$

Common Mistakes:

- Some students did not know that the set of natural numbers $\mathbb{N}$ does not include negative numbers. Therefore, many students gave $x = -1$ as a counterexample. Observe that if the domain included negative numbers, then f would not be a function at all since it is undefined for $x = -1$.
- Many students erroneously claimed that f is indeed a bijection and provided a spurious "proof" of surjectivity.

(c) (10 points) Let f be a surjective function from B to C, and let g be a surjective function from A to B. Prove that $f \circ g$ is also surjective. Your answer should clearly state any proof strategies used in the proof.

Solution:

A function is surjective if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$. We will use a proof by contradiction strategy to prove that $f \circ g$ is surjective. For contradiction, suppose that there exists an element $c \in C$ such that $\forall x.\ (f \circ g)(x) \neq c$. Now, by definition of composition, this implies:

$$\forall x.\ f(g(x)) \neq c$$

But now, since f is surjective, there must exist some b in B such that $f(b) = c$. Furthermore, since g is also surjective, there must exist some $a \in A$ such that $g(a) = b$. Since $g(a) = b$ and $f(b) = c$, we have $(f \circ g)(a) = f(g(a)) = c$. But this contradicts our assumption that $\forall x.\ (f \circ g)(x) \neq c$.

Common Mistakes: Very few students (maybe 2) answered this question correctly. Since so few students answered the question correctly and since the original question contained a small typo, this question was not taken into account when assigning grades for the midterm. Therefore, the midterm score is out of 80 instead of out of 90.

The typo in the original question was that it states f is from A to B instead of B to C, and that g is from B to C instead of from A to B.

5. Prove or disprove the claims below. If you believe the claim is true, prove your answer and explicitly name the proof strategies you use. If you believe the claim is false, provide a counterexample.

   (a) (9 points) Prove or disprove: If $x^2 - 6x + 5$ is even, then x is odd.

   Solution:

   This statement is correct, and we will prove it using proof by contraposition. First, observe that the contrapositive of this statement is "If x is even, then $x^2 - 6x + 5$ is odd." Now, assume x is even. Then, there exists some integer k such that $x = 2k$. Then, $x^2 - 6x + 5 = 4k^2 - 12k + 5 = 2(2k^2 - 6k + 2) + 1$. Since $x^2 - 6x + 5$ can be written as $2k' + 1$ for some integer k', it follows that it is odd.

(b) (6 points) Prove or disprove the following claim: Given two sets A and B, $A \times B = \emptyset$ if and only if $A = \emptyset$ and $B = \emptyset$

Solution:

This claim is wrong. $A \times B = \emptyset$ if either $A = \emptyset$ *or* $B = \emptyset$. For instance, $\{1\} \times \emptyset = \emptyset$

Common Mistakes: Many students erroneously claimed that $\emptyset \times \{1\} = \{(\emptyset, 1)\}$. The Cartesian product of any set with the empty set yields the empty set.