CS311H Problem Set 4

Due Tuesday, October 16

Please hand in a hard copy of your solutions before class on the due date. You may discuss problems with other students in the class; however, your write-up must mention the names of these individuals.

1. (20 points) Prove or disprove each of the following claims:

   (a) The function f defined by $f(x) = 2x + 5$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

   (b) The function f defined by $f(x) = 2x + 5$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$.

2. (15 points) Let x be a real number. Prove the following:

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$$

3. (10 points) State whether each of the statements below are true or false. Briefly justify your answer.

   (a) Let A be a countable set and B be a countably infinite set. Then, $A \cup B$ is always countable.

   (b) The set $\mathbb{R} - \mathbb{N}$ is countably infinite.

   (c) Let A and B be two uncountably infinite sets. Then $A \cap B$ is uncountable.

4. (10 points) Let a, b, c, m be integers such that $m \geq 2$ and $c > 0$. Prove or find a counterexample to the following claim: "If $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{cm}$."

5. (10 points) Let a, b, m be integers such that $m \geq 2$. Prove or find a counterexample to the following claim: "If $a \equiv b \pmod{m}$, then $\gcd(a, m) = \gcd(b, m)$."

6. (10 points) Use the extended Euclidian algorithm to find $\gcd(215, 35)$, and find integers s, t such that $\gcd(215, 35) = s \cdot 215 + t \cdot 35$. Show every step of the algorithm.

7. (3 points) Determine if the linear congruence $12x \equiv 15 \pmod 8$ has any solutions. Explain your reasoning.

8. (7 points) Find the set of all solutions to the linear congruence $3x \equiv 4 \pmod 7$. Show all your work.

9. (10 points) For any integer $n \geq 1$, prove that there always exist n consecutive composite numbers. (Hint: You might want to use the factorial function.)