CS311H: Discrete Mathematics

Asymptotic Analysis

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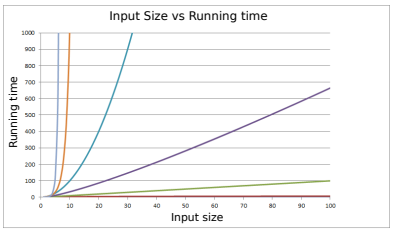
Complexity Theory Intro

- Algorithmic complexity theory: How fast does the running time of an algorithm grow with respect to input size?
- Some algorithms scale better as input size grows



Complexity Theory

- Complexity theory is concerned with expressing the running time of an algorithm with respect to input size
- Idea 1: Express running time in terms of the number of operations – abstract away choice of hardware and PL
- Idea 2: Not interested in exact number of operations, just an asymptotic estimate
  - Number of operations proportional to some function $f(n)$ s.t. they become equal as $n \rightarrow \infty$
  - e.g., instead of $2 \cdot n^3 + \frac{1}{2}n^2 + \frac{11}{9}n + 4log(n)$, we just say running time is cubic

Big-O Notation

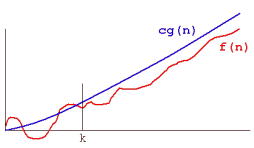
- Useful tool for asymptotic analysis is Big-O notation
- Intuition: If algorithm is $O(f(n))$, then running time is bounded by a function proportional to $f(n)$ for sufficiently large n
- Example: If algorithm is $O(1)$, this means it's constant-time, i.e., running time of algorithm does not depend on input size
- Example: If algorithm is $O(n)$, running time of algorithm grows linearly in input size

Formal Definition of Big-O

A function $f(n)$ is $O(g(n))$ if there exist positive constants C, k such that:

$$\forall n > k.\ |f(n)| \leq C|g(n)|$$

- i.e., for sufficiently large n, $f(n)$ is bounded from above by a function proportional to $g(n)$



An Assumption in this Course

A function $f(n)$ is $O(g(n))$ if there exist positive constants C, k such that:

$$\forall n > k.\ |f(n)| \leq C|g(n)|$$

- In the context of algorithmic complexity, $f(n)$ represents the number of operations performed
- Hence, both $f(n)$ and $g(n)$ will always be positive – you can assume this for this class, so we'll omit the absolute value

Example I

Prove that $4n + 2$ is $O(n)$

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Example II

Prove that $\frac{1}{2}n^2$ is not $O(n)$

- Proof by contradiction – suppose there was some C, k s.t.:

$$\forall n > k.\ \frac{1}{2}n^2 \leq Cn$$

- Implies $\forall n > k.\ n \leq 2C$
- But this is not true because k, C are constants and we can make n arbitrarily large

Example III

Prove that $\log_2(3n^2 + 1)$ is $O(\log_2 n)$

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Useful Theorem

Let $f(n)$ be a polynomial of degree d. Then $f(n)$ is $O(n^d)$.

- $f(n)$ must be $a_d n^d + a_{d-1} n^{d-1} + \ldots + a_0$
- Observe: $f(n) = n^d(a_d + \frac{a_{d-1}}{n} + \ldots \frac{a_0}{n^d})$
- Since $n > 1$, $f(n) \leq n^d(a_d + a_{d-1} + \ldots + a_0)$
- Now, pick $C = a_d + a_{d-1} + \ldots + a_0$, and $k = 1$
- Hence, $\forall n > 1.\ f(n) \leq C \cdot n^d$

Growth of Combination of Functions

- We are often interested in understanding the combined growth of multiple functions
- Consider a procedure `P(n) = foo(n); bar(n);`
- If we know `foo` is $O(f(n))$ and `bar` is $O(g(n))$, what can we say about P?

Growth of Sum of Two Functions

Suppose $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then, $(f_1 + f_2)(x)$ is $O(\max(g_1(x), g_2(x)))$

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Example

- Find a Big-O estimate for $f(n) = log(3n^2 + 1) + \frac{11}{6}n^2$
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Another Example

- Consider a procedure `P(n) = foo(n); bar(n);`
- If complexity of `foo` is $O(n)$ and that of `bar` is $O(n \log n)$, what can we say about complexity of `P`?
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Another Useful Theorem

Suppose $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then, $(f_1 \cdot f_2)(x)$ is $O((g_1(x) \cdot g_2(x)))$

- We know $\exists C_1, C_2, k_1, k_2$ such that:

$$\forall n > k_1.\ f_1(x) \leq C_1 \cdot g_1(x)$$
$$\forall n > k_2.\ f_2(x) \leq C_2 \cdot g_2(x)$$

- Let $k = \max(k_1, k_2)$ and $C = C_1 C_2$
- Hence, $\forall n > k.\ (f_1 \cdot f_2)(x) \leq C \cdot g_1(x) \cdot g_2(x)$
- Thus, $(f_1 \cdot f_2)(x)$ is $O(g_1(x) \cdot g_2(x))$

Example

- Find a Big-O estimate for $f(n) = (\frac{1}{2}n^2 + 5)(log(3n^2 + 1))$
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Another example

- Assuming complexity of `g` is $O(\log n)$, find a Big-O estimate for the following procedure `f`:

```
f(n) = { for(i = 0; i < n; i+=2) { g(i, n) } }
```

- Total # of operations = # of times loop executes * # of operations performed by g
- What is big-O estimate for the loop?
- Big-O estimate for f:

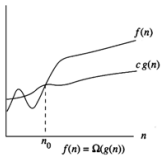
Big-Omega Notation

- Big-O notation is useful for giving an upper bound for $f(n)$ for large values of n
- But sometimes we are also interested in a lower bound!
- For this purpose, we use the Big-Omega Ω notation, which represents asymptotic lower bounds

Formal Definition of Big-Omega

A function $f(n)$ is $\Omega(g(n))$ if there exist positive constants C, k such that:

$$\forall n > k.\ |f(n)| \geq C|g(n)|$$



- i.e., for sufficiently large n, $f(n)$ is bounded from below by a function proportional to $g(n)$
- As before, we will assume $f(n)$ and $g(n)$ are positive

Example

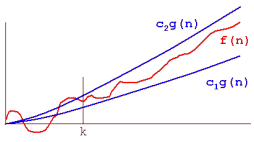
1. Prove that $2n^2 + n$ is $\Omega(n)$
   - Find a C and k:
   -
2. Prove that $5n + 1$ is not $\Omega(n^2)$
   - Suppose there exists constants such that:

     $$\forall n > k.\ 5n + 1 \geq C \cdot n^2$$

   - Implies $\forall n > k.\ C \cdot n^2 - 5n - 1 \leq 0$
   - But since $C > 0$, $C \cdot n^2 - 5n - 1$ is an upward-looking looking parabola $\Rightarrow$ for n large enough, $C \cdot n^2 - 5n - 1$ is positive

Big Theta

- So far: Big-O gives asymptotic upper bounds, and Big-Omega gives asymptotic lower bounds
- But sometimes we are intereted in a function that serves both as an asymptotic lower and upper bound



- This is expressed using Big-Theta notation

Formal Definition of Big-Theta

- A function $f(x)$ is said to be $\Theta(g(x))$ if $f(x)$ is both $\Omega(g(x))$ and $O(g(x))$
- Another way of saying this:

  $$\exists C_1, C_2, k > 0.\ \forall n > k.\ C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$$

Example I

Prove that $10n + 4^{\log_2 n}$ is $\Theta(n^2)$

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Example II

Prove that $\log n! = \Theta(n\ \log n)$

- First show $\log n!$ is $O(n\ \log n)$:

Example II, cont.

- Now show that $\log\ n!$ is $\Omega(n\ \log n)$:

A Useful Theorem

1. $f(x)$ is $O(g(x))$ if:

$$\lim_{x\to\infty} |\frac{f(x)}{g(x)}| < \infty$$

2. $f(x)$ is $\Omega(g(x))$ if:

$$\lim_{x\to\infty} |\frac{f(x)}{g(x)}| > 0$$

3. $f(x)$ is $\Theta(g(x))$ if:

$$0 < \lim_{x\to\infty} |\frac{f(x)}{g(x)}| < \infty$$

Example I

- Show that $\frac{x^2+1}{x+1}$ is $\Theta(x)$
- Use previous theorem:

$$\lim_{x\to\infty} \frac{x^2+1}{x+1} \cdot \frac{1}{x} = 1$$

- Hence, it's case (3) $\Rightarrow \Theta(x)$

Example II

- Consider $f(n) = 2^n$ and $g(n) = 3^n$.
- Which of the following are true about $f(n)$?
  - It's $O(g(n))$
  - It's $\Omega(g(n))$
  - It's $\Theta(g(n))$

Example III

- Consider $f(n) = \log_2 n$ and $g(n) = 2\ \log_3 n$.
- Which of the following are true about $f(n)$?
  - It's $O(g(n))$
  - It's $\Omega(g(n))$
  - It's $\Theta(g(n))$