CS311H: Discrete Mathematics

Permutations and Combinations

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Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects
  - No object can be selected more than once
  - Order of arrangement matters
- Example: $S = \{a, b, c\}$. What are the permutations of S?
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How Many Permutations?

- Consider set $S = \{a_1, a_2, \ldots a_n\}$
- How many permutations of S are there?
- Decompose using product rule:
  - How many ways to choose first element?
  - How many ways to choose second element?
  - ...
  - How many ways to choose last element?
- What is number of permutations of set S?

Examples

- Consider the set $\{7, 10, 23, 4\}$. How many permutations?
- How many permutations of letters A, B, C, D, E, F, G contain "ABC" as a substring?
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r-Permutations

- r-permutation is ordered arrangment of r elements in a set S
  - S can contain more than r elements
  - But we want arrangement containing r of the elements in S
- The number of r-permutations in a set with n elements is written $P(n, r)$
- Example: What is $P(n, n)$?

Computing $P(n, r)$

- Given a set with n elements, what is $P(n, r)$?
- Decompose using product rule:
  - How many ways to pick first element?
  - How many ways to pick second element?
  - How many ways to pick i'th element?
  - How many ways to pick last element?
- Thus, $P(n, r) = n \cdot (n-1) \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$

Examples

- What is the number of 2-permutations of set $\{a, b, c, d, e\}$?
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- How many ways to select first-prize winner, second-prize winner, third-prize winner from 10 people in a contest?
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- Salesman must visit 4 cities from list of 10 cities: Must begin in Chicago, but can choose the remaining cities and order.
- How many possible itinerary choices are there?
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Combinations

- An r-combination of set S is the unordered selection of r elements from that set
  - Unlike permutations, order does not matter in combinations
- Example: What are 2-combinations of the set $\{a, b, c\}$?
- For this set, 6 2-permutations, but only 3 2-combinations

Number of r-combinations

- The number of r-combinations of a set with n elements is written $C(n, r)$
- $C(n, r)$ is often also written as $\binom{n}{r}$, read "n choose r"
- $\binom{n}{r}$ is also called the binomial coefficient
- Theorem:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r! \cdot (n - r)!}$$

Proof of Theorem

- What is the relationship between $P(n, r)$ and $C(n, r)$?
- Let's decompose $P(n, r)$ using product rule:
  - First choose r elements
  - Then, order these r elements
- How many ways to choose r elements from n?
- How many ways to order r elements?
- Thus, $P(n, r) = C(n, r) * r!$
- Therefore,

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n - r)! \cdot r!}$$

Examples

- How many hands of 5 cards can be dealt from a standard deck of 52 cards?
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- There are 9 faculty members in a math department, and 11 in CS department.
- If we must select 3 math and 4 CS faculty for a committee, how many ways are there to form this committee?
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More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?
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One More Example

- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?
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Binomial Coefficients

- Recall: $C(n, r)$ is also denoted as $\binom{n}{r}$ and is called the binomial coefficient
- Binomial is polynomial with two terms, e.g., $(a + b), (a + b)^2$
- $\binom{n}{r}$ called binomial coefficient b/c it occurs as coefficients in the expansion of $(a + b)^n$

An Example

- Consider expansion of $(a + b)^3$
- $(a + b)^3 = (a + b)(a + b)(a + b)$
- $= (a^2 + 2ab + b^2)(a + b)$
- $= (a^3 + 2a^2b + ab^2) + (a^2b + 2ab^2 + b^3)$
- $= 1a^3 + 3a^2b + 3ab^2 + 1b^3$

1 **3** **3** **1**

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

The Binomial Theorem

- Let x, y be variables and n a non-negative integer. Then,

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

- What is the expansion of $(x + y)^4$?

Another Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
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Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.
- Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
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Another Corollary

- Corollary: $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$
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Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

- Pascal arranged binomial coefficients as a triangle
- n'th row consists of $\binom{n}{k}$ for $k = 0, 1, \ldots n$

Proof of Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- This identity is known as Pascal's identity
- Proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!}$$

- Multiply first fraction by $\frac{k}{k}$ and second by $\frac{n-k+1}{n-k+1}$:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

Proof of Pascal's Identity, cont.

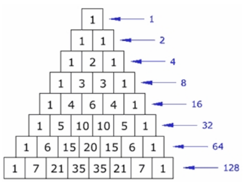
$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

- Factor the numerator:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{(n+1) \cdot n!}{(k)!(n-k+1)!} = \frac{(n+1)!}{k! \cdot (n-k+1)!}$$

- But this is exactly $\binom{n+1}{k}$

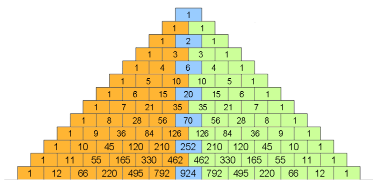
Interesting Facts about Pascal's Triangle



- What is the sum of numbers in n'th row in Pascal's triangle (starting at $n = 0$)?
- Observe: This is exactly the corollary we proved earlier!

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Some Fun Facts about Pascal's Triangle, cont.



- Pascal's triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
- Why is this the case?

Permutations with Repetitions

- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement
- But sometimes makes sense to use an object multiple times
- Example: How many strings of length 4 can be formed using letters in English alphabet?
- Since string can contain same letter multiple times, we want to allow repetition!
- A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once

General Formula for Permutations with Repetition

- $P^*(n, r)$ denotes number of r-permutations with repetition from set with n elements
- What is $P^*(n, r)$?
- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different 3-digit numbers can be formed from $1, 2, 3, 4, 5$?

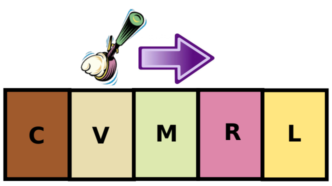
Combinations with Repetition

- Combinations help us to answer the question "In how many ways can we choose r objects from n objects?"
- Now, consider the slightly different question: "In how many ways can we choose r objects from n kinds of objects?
- These questions are quite different:
  - For first question, once we pick one of the n objects, we cannot pick the same object again
  - For second question, once we pick one of the n kinds of objects, we can pick the same type of object again!
- Combination with repetition allows answering the latter type of question!

Example

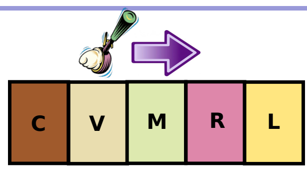
- An ice cream dessert consists of three scoops of ice cream
- Each scoop can be one of the flavors: chocolate, vanilla, mint, lemon, raspberry
- In how many different ways can you pick your dessert?
- Example of combination with repetition: "In how many ways can we pick 3 objects from 5 kinds of objects?"
- Caveat: Despite looking deceptively simple, quite difficult to figure this out (at least for me...)

Example, cont.



- To solve problem, imagine we have ice cream in boxes.
- We start with leftmost box, and proceed towards right.
- At every box, you can take 0-3 scoops, and then move to next.
- Denote taking a scoop by $\circ$ and moving to next box by $\rightarrow$

Example, cont.



- Let's look at some selections and their representation:
  - 3 scoops of chocolate: $\circ \circ \circ \rightarrow\rightarrow\rightarrow\rightarrow$
  - 1 vanilla, 1 raspberry, 1 lemon: $\rightarrow \circ \rightarrow\rightarrow \circ \rightarrow \circ$
  - 2 mint, 1 raspberry: $\rightarrow\rightarrow \circ\circ \rightarrow \circ \rightarrow$
- Invariant: r circles and $n-1$ arrows (here, $r = 3, n = 5$)
- Our question is equivalent to: "In how many ways can we arrange r circles and $n-1$ arrows?"

Result

- We'll denote the number of ways to choose r objects from n kinds of objects $C^*(n,r)$:

$$C^*(n,r) = \binom{n+r-1}{r}$$

- Example: In how many ways can we choose 3 scoops of ice cream from 5 different flavors?
- Here, $r = 3$ and $n = 5$. Thus:

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = 35$$

Example 1

- Suppose there is a bowl containing apples, oranges, and pears
  - There is at least four of each type of fruit in the bowl
- How many ways to select four pieces of fruit from this bowl?
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Example 2

- Consider a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills
  - There is at least five of each type of bill in the box
- How many ways are there to select 5 bills from this cash box?
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Example 3

- Assuming x_1, x_2, x_3 are non-negative integers, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Example 4

- Suppose x_1, x_2, x_3 are integers s.t. $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.
- Then, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Summary of Different Permuations and Combinations

Order matters?	Question: How many ways to pick r objects from ...	
	n objects	n types of objects
Yes	Permutation $P(n,r) = \frac{n!}{(n-r)!}$	Permutation w/ repetition $P^*(n,r) = n^r$
No	Combination $C(n,r) = \frac{n!}{r!\cdot(n-r)!}$	Combination w/ repetition $C^*(n,r) = \frac{(n+r-1)!}{r!\cdot(n-1)!}$