CS311H: Discrete Mathematics

Permutations and Combinations

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Review

- What is $P(n, r)$?
- What is $C(n, r)$?
- What is the inclusion-exclusion principle?

More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?

One More Example

- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?

Binomial Coefficients

- Recall: $C(n, r)$ is also denoted as $\binom{n}{r}$ and is called the binomial coefficient
- Binomial is polynomial with two terms, e.g., $(a + b), (a + b)^2$
- $\binom{n}{r}$ called binomial coefficient b/c it occurs as coefficients in the expansion of $(a + b)^n$

An Example

- Consider expansion of $(a + b)^3$
- $(a + b)^3 = (a + b)(a + b)(a + b)$
- $= (a^2 + 2ab + b^2)(a + b)$
- $= (a^3 + 2a^2b + ab^2) + (a^2b + 2ab^2 + b^3)$
- $= 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$$\begin{array}{cccc} \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{array}$$

The Binomial Theorem

- Let x, y be variables and n a non-negative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

- What is the expansion of $(x+y)^4$?

Another Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
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Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.
- Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
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Another Corollary

- Corollary: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
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Pascal's Triangle

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Proof of Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- This identity is known as Pascal's identity
- Proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!}$$

- Multiply first fraction by $\frac{k}{k}$ and second by $\frac{n-k+1}{n-k+1}$:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

Proof of Pascal's Identity, cont.

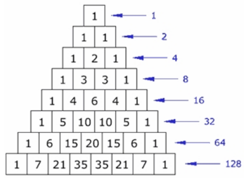
$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

- Factor the numerator:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{(n+1)\cdot n!}{(k)!(n-k+1)!} = \frac{(n+1)!}{k!\cdot(n-k+1)!}$$

- But this is exactly $\binom{n+1}{k}$

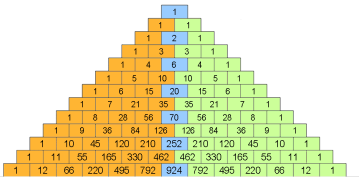
Interesting Facts about Pascal's Triangle



- What is the sum of numbers in n'th row in Pascal's triangle (starting at $n = 0$)?
- Observe: This is exactly the corollary we proved earlier!

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Some Fun Facts about Pascal's Triangle, cont.



- Pascal's triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
- Why is this the case?

Permutations with Repetitions

- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement
- But sometimes makes sense to use an object multiple times
- Example: How many strings of length 4 can be formed using letters in English alphabet?
- Since string can contain same letter multiple times, we want to allow repetition!
- A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once

General Formula for Permutations with Repetition

- $P^*(n, r)$ denotes number of r-permutations with repetition from set with n elements
- What is $P^*(n, r)$?
- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different 3-digit numbers can be formed from $1, 2, 3, 4, 5$?

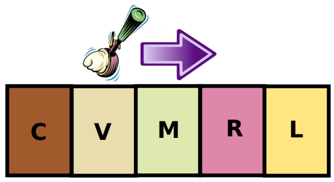
Combinations with Repetition

- Combinations help us to answer the question "In how many ways can we choose r objects from n objects?"
- Now, consider the slightly different question: "In how many ways can we choose r objects from n kinds of objects?
- These questions are quite different:
  - For first question, once we pick one of the n objects, we cannot pick the same object again
  - For second question, once we pick one of the n kinds of objects, we can pick the same type of object again!
- Combination with repetition allows answering the latter type of question!

Example

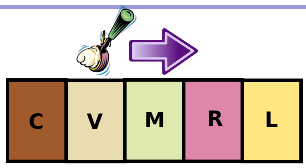
- An ice cream dessert consists of three scoops of ice cream
- Each scoop can be one of the flavors: chocolate, vanilla, mint, lemon, raspberry
- In how many different ways can you pick your dessert?
- Example of combination with repetition: "In how many ways can we pick 3 objects from 5 kinds of objects?"
- Caveat: Despite looking deceptively simple, quite difficult to figure this out (at least for me...)

Example, cont.



- To solve problem, imagine we have ice cream in boxes.
- We start with leftmost box, and proceed towards right.
- At every box, you can take 0-3 scoops, and then move to next.
- Denote taking a scoop by $\circ$ and moving to next box by $\rightarrow$

Example, cont.



- Let's look at some selections and their representation:
  - 3 scoops of chocolate: $\circ \circ \circ \rightarrow\rightarrow\rightarrow\rightarrow$
  - 1 vanilla, 1 raspberry, 1 lemon: $\rightarrow \circ \rightarrow\rightarrow \circ \rightarrow \circ$
  - 2 mint, 1 raspberry: $\rightarrow\rightarrow \circ\circ \rightarrow \circ \rightarrow$
- Invariant: r circles and $n-1$ arrows (here, $r=3, n=5$)
- Our question is equivalent to: "In how many ways can we arrange r circles and $n-1$ arrows?"

Result

- We'll denote the number of ways to choose r objects from n kinds of objects $C^*(n,r)$:

$$C^*(n,r) = \binom{n+r-1}{r}$$

- Example: In how many ways can we choose 3 scoops of ice cream from 5 different flavors?
- Here, $r=3$ and $n=5$. Thus:

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = 35$$

Example 1

- Suppose there is a bowl containing apples, oranges, and pears
  - There is at least four of each type of fruit in the bowl
- How many ways to select four pieces of fruit from this bowl?
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Example 2

- Consider a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills
  - There is at least five of each type of bill in the box
- How many ways are there to select 5 bills from this cash box?
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Example 3

- Assuming x_1, x_2, x_3 are non-negative integers, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Example 4

- Suppose x_1, x_2, x_3 are integers s.t. $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.
- Then, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Summary of Different Permuations and Combinations

Order matters?	Question: How many ways to pick r objects from ...	
	n objects	n types of objects
Yes	Permutation $P(n, r) = \frac{n!}{(n-r)!}$	Permutation w/ repetition $P^*(n, r) = n^r$
No	Combination $C(n, r) = \frac{n!}{r! \cdot (n-r)!}$	Combination w/ repetition $C^*(n, r) = \frac{(n+r-1)!}{r! \cdot (n-1)!}$

Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters in the word ALL?
- This is not given by $3!$ because some of the letters in this word are the same
- Different strings: ALL, LAL, LLA $\Rightarrow$ 3 possibilities, not 6 because relative ordering of L's doesn't matter
- In general, how can we compute the number of permutations of n objects where some of them are indistinguishable?

Permutations with Indistinguishable Objects, cont.

- Consider n objects such that:
  - n_1 of them indistingishable of type 1
  - n_2 of them indistingishable of type 2
  - ...
  - n_k of them indistinguishable of type k
- The number of permutations in this case is given by:

$$\frac{n!}{n_1! n_2! \ldots n_k!}$$

Proof

- Let's decompose using product rule:
  - First, place all n_1 objects of type 1
  - Then all n_2 objects of type 2 etc.
- How many ways to place n_1 indistinguishable objects in n slots?
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Proof, cont.

- Now, how many ways to place n_2 objects of type 2?
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- Continuing this way and using product rule, number of permutations is:

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdots\binom{n-\sum_{i=1}^{k-1} n_i}{n_k}$$

Proof, cont.

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdots\binom{n-\sum_{i=1}^{k-1} n_i}{n_k}$$

- Let's expand this definition:

$$\frac{n!}{n_1!\cdot(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\cdot(n-n_1-n_2)!}\cdot\ldots\cdot\frac{(n-n_1-n_2-\ldots n_{k-1})!}{k!\cdot 0!}$$

$$\frac{n!}{n_1!\cdot\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\cdot\cancel{(n-n_1-n_2)!}}\cdot\ldots\cdot\frac{\cancel{(n-n_1-n_2-\ldots n_{k-1})!}}{k!\cdot 0!}$$

- This simplifies to: $\dfrac{n!}{n_1!n_2!\ldots n_k!}$
- Another way to see this: Compute total # of permutations ($n!$) and then divide by # of relative orderings between objects of type 1 ($n_1!$), # of relative orderings of objects of type 2 ($n_2!$) etc.

Example 1

- How many different strings can be made by ordering the letters of the word SUCCESS?
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Example 2

- There are 3 identical red balls, 5 identical blue balls, and 2 identical green balls.
- In how many different ways can these balls be arranged if the first ball must be blue?
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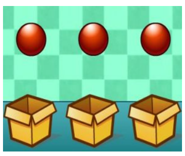
Distributing Objects into Boxes



- Many counting problems can be thought of as distributing objects into boxes
- In some cases both objects and boxes are distinguishable
- In some cases, boxes are distinguishable, but objects are indistinguishable

Example: Distinguishable Objects in Distinguishable Boxes

- How many ways are there to distribute 5 cards to each of 4 players from a deck of 52 cards?
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Example, cont.

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Indistinguishable Objects Into Distinguishable Boxes

- How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?
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Distributing Objects into Boxes Summary

- Distinguishable objects into distinguishable boxes:
  - Involves permutations
- Indistinguishable objects into distinguishable boxes:
  - Involves combination

Another Example

- How many ways to distribute six distinguishable objects to five distinguishable boxes?
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Example 2

- How many ways to distribute six indistinguishable objects to five distinguishable boxes?
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Example 3

- How many ways to assign 15 distinguishable objects into 5 distinguishable boxes so boxes contain $1, 2, 3, 4, 5$ objects respectively?
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