CS311H: Discrete Mathematics

Functions

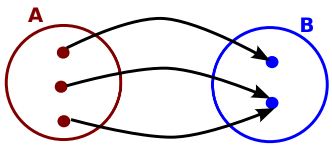
Instructor: Işıl Dillig

Functions

- A function f from a set A to a set B assigns each element of A to exactly one element of B.
- A is called domain of f, and B is called codomain of f.
- If f maps element $a \in A$ to element $b \in B$, we write $f(a) = b$
- If $f(a) = b$, b is called image of a; a is in preimage of b.
- Range of f is the set of all images of elements in A.

Functions Examples and Non-Examples

Is this mapping a function?



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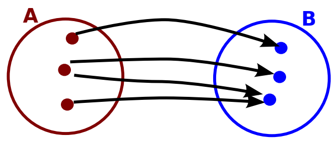

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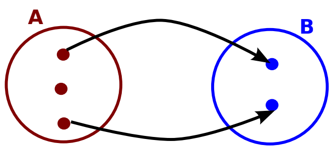

Functions Examples and Non-Examples

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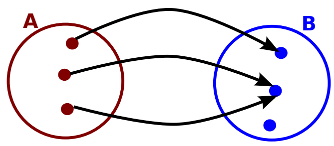
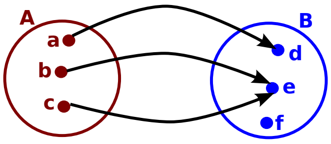


Function Terminology Examples



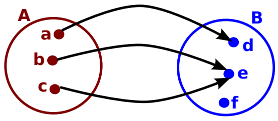
- What is the range of this function?
- What is the image of c?
- What is the preimage of e?

Image of a Set

- We can extend the definition of image to a set
- Suppose f is a function from A to B and S is a subset of A
- The image of S under f includes exactly those elements of B that are images of elements of S:

$$f(S) = \{t \mid \exists s \in S.\ t = f(s)\}$$

- What is the image of $\{b, c\}$?



One-to-One Functions

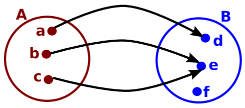
- A function f is called one-to-one if and only if $f(x) = f(y)$ implies $x = y$ for every x, y in the domain of f:

$$\forall x, y.\ (f(x) = f(y) \rightarrow x = y)$$

- One-to-one functions never assign different elements in the domain to the same element in the codomain:

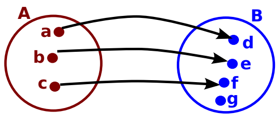
$$\forall x, y.\ (x \neq y \rightarrow f(x) \neq f(y))$$

- A one-to-one function also called injection or injective function
- Is this function one-to-one?



More Injective Function Examples

- Is this function injective?



- Consider the function $f(x) = x^2$ from set of integers to set of integers. Is this injective?
- What about if the domain of f is the set of non-negative integers?

Proving Injectivity Example

- Consider the function f from $\mathbb{Z}$ to $\mathbb{Z}$ defined as:

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 0 \\ -3x + 2 & \text{if } x < 0 \end{cases}$$

- Prove that f is injective.
- We need to show that if $x \neq y$, then $f(x) \neq f(y)$
- What proof technique do we need to use?

Proving Injectivity Example, cont.

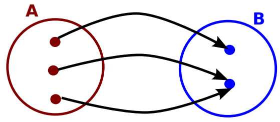
Proving Injectivity Example, cont.

Onto Functions

- A function f from A to B is called onto iff for every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$:

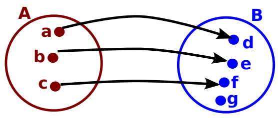
$$\forall y \in B. \exists x \in A.\ f(x) = y$$

- Note: $\exists x \in A.\ \phi$ is shorthand for $\exists x.(x \in A \wedge \phi)$, and $\forall x \in A.\ \phi$ is shorthand for $\forall x.(x \in A \rightarrow \phi)$
- Onto functions also called surjective functions or surjections
- For onto functions, range and codomain are the same
- Is this function onto?



Examples of Onto Functions

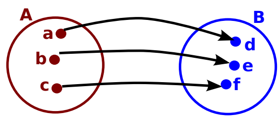
- Is this function onto?



- Consider the function $f(x) = x^2$ from the set of integers to the set of integers. Is f surjective?

Bijective Functions

- Function that is both onto and one-to-one called bijection
- Bijection also called one-to-one correspondence or invertible function
- Example of bijection:



Bijection Example

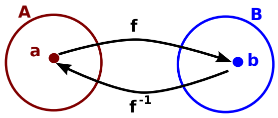
- The identity function I on a set A is the function that assigns every element of A to itself, i.e., $\forall x \in A.\ I(x) = x$
- Prove that the identity function is a bijection.
- Need to prove I is both one-to-one and onto.
- One-to-one: We need to show $\forall x, y.\ (x \neq y \rightarrow I(x) \neq I(y))$
- Suppose $x \neq y$.
- Since $I(x) = x$ and $I(y) = y$, and $x \neq y$, $I(x) \neq I(y)$

Bijection Example, cont.

- Now, prove I is onto, i.e., for every b, there exists some a such that $f(a) = b$
- For contradiction, suppose there is some b such that $\forall a \in A.\ I(a) \neq b$
- Since $I(a) = a$, this means $\forall a \in A.\ a \neq b$
- But since b is itself in A, this would imply $b \neq b$, yielding a contradiction.
- Since I is both onto and one-to-one, it is a bijection. $\square$

Inverse Functions

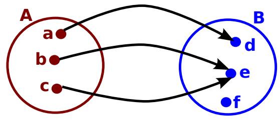
- Every bijection from set A to set B also has an inverse function
- The inverse of bijection f, written f^{-1}, is the function that assigns to $b \in B$ a unique element $a \in A$ such that $f(a) = b$



- Observe: Inverse functions are only defined for bijections, not arbitrary functions!
- This is why bijections are also called invertible functions

Why are Inverse Functions Only Defined on Bijections?

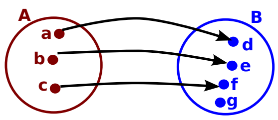
- Suppose f is not injective, i.e., assigns distinct elements to the same element.



- Then, the inverse is not a function because it assigns the same element to distinct elements

Why are Inverse Functions Only Defined on Bijections?

- Suppose f is not surjective, i.e., range and codomain are not the same



- Then, the inverse is not a function because it does not assign some element in B to any element in A
- Hence, inverse functions only defined for bijections!

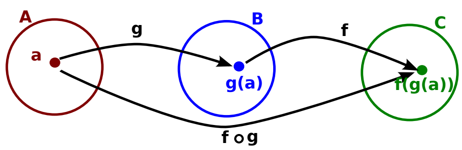
Inverse Function Examples

- Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = x^2$. Is f invertible?
- Let g be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $g(x) = x + 1$. Is g invertible?

Function Composition

- Let g be a function from A to B, and f from B to C.
- The composition of f and g, written $f \circ g$, is defined by:

$$(f \circ g)(x) = f(g(x))$$



Composition Example

- Let f and g be function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = 2x + 3$ and $g(x) = 3x + 2$
- What is $f \circ g$?

Another Composition Example

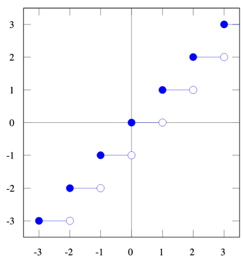
- Prove that $f^{-1} \circ f = I$ where I is the identity function.
- Since $I(x) = x$, need to show $(f^{-1} \circ f)(x) = x$
- First, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
- Let $f(x)$ be y
- Then, $f^{-1}(f(x)) = f^{-1}(y)$
- By definition of inverse, $f^{-1}(y) = x$ iff $f(x) = y$
- Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ $\square$

Example

- Prove that if f and g are injective, then $f \circ g$ is also injective.

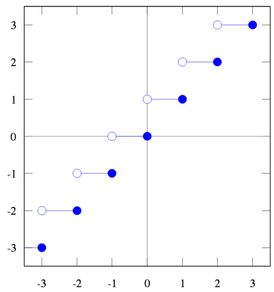
Floor and Ceiling Functions

- Two important functions in discrete math are floor and ceiling functions, both from $\mathbb{R}$ to $\mathbb{Z}$
- The floor of a real number x, written $\lfloor x \rfloor$, is the largest integer less than or equal to x.

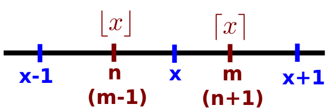


Ceiling Function

- The ceiling of a real number x, written $\lceil x \rceil$, is the smallest integer greater than or equal to x.

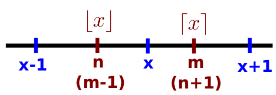


Useful Properties of Floor and Ceiling Functions



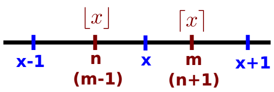
1. For integer n and real number x, $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$
2. For integer n and real number x, $\lceil x \rceil = m$ iff $m - 1 < x \leq m$
3. For any real x, $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

Proofs about Floor/Ceiling Functions



Prove that $\lfloor -x \rfloor = -\lceil x \rceil$

Another Example



Prove that $\lfloor x + k \rfloor = \lfloor x \rfloor + k$ where k is an integer

More Examples

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- Observe: Any real number x can be written as $n + \epsilon$ where $n = \lfloor x \rfloor$ and $0 \leq \epsilon < 1$
- To prove desired property, do proof by cases
- Case 1: $0 \leq \epsilon < \frac{1}{2}$
- Case 2: $\frac{1}{2} \leq \epsilon < 1$
- First prove property for first case, then second case

Revisiting Sets

- Earlier we talked about sets and cardinality of sets
- Recall: Cardinality of a set is number of elements in that set
- This definition makes sense for sets with finitely many element, but more involved for infinite sets
- Agenda: Revisit the notion of cardinality for infinite sets

Cardinality of Infinite Sets

- Sets with infinite cardinality are classified into two classes:
  1. Countably infinite sets (e.g., natural numbers)
  2. Uncountably infinite sets (e.g., real numbers)
- A set A is called countably infinite if there is a bijection between A and the set of positive integers.
- A set A is called countable if it is either finite or countably infinite
- Otherwise, the set is called uncountable or uncountably infinite

Example

Prove: The set of odd positive integers is countably infinite.

- Need to find a function f from $\mathbb{Z}^+$ to the set of odd positive integers, and prove that f is bijective
- Consider $f(n) = 2n - 1$ from $\mathbb{Z}^+$ to odd positive integers
- We need to show f is bijective (i.e., one-to-one and onto)
- Let's first prove injectivity, then surjectivity

Example, cont.

Another Way to Prove Countable-ness

- One way to show a set A is countably infinite is to give bijection between $\mathbb{Z}^+$ and A
- Another way is by showing members of A can be written as a sequence $(a_1, a_2, a_3, \ldots)$
- Since such a sequence is a bijective function from $\mathbb{Z}^+$ to A, writing A as a sequence $a_1, a_2, a_3, \ldots$ establishes one-to-one correspondence

Another Example

Prove that the set of all integers is countable

- We can list all integers in a sequence, alternating positive and negative integers:

$$a_n = 0, 1, -1, 2, -2, 3, -3, \ldots$$

- Observe that this sequence defines the bijective function:

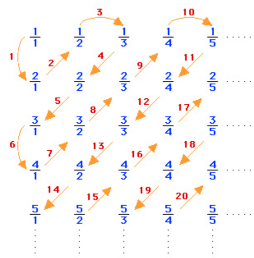
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n-1)/2 & \text{if } n \text{ odd} \end{cases}$$

Rational Numbers are Countable

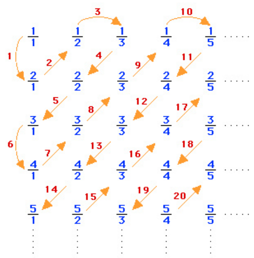
- Not too surprising $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite
- More surprising: Set of rationals is also countably infinite!
- We'll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence
- Recall: Every positive rational number can be written as the quotient p/q of two positive integers p, q

Rationals in a Table

- Now imagine placing rationals in a table such that:
  1. Rationals with $p = 1$ go in first row, $p = 2$ in second row, etc.
  2. Rationals with $q = 1$ in 1st column, $q = 2$ in 2nd column, ...

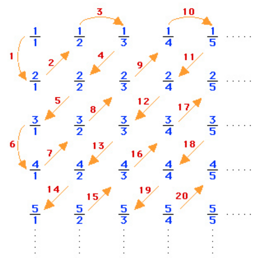


Enumerating the Rationals



- How to enumerate entries in this table without missing any?
- Trick: First list those with $p + q = 2$, then $p + q = 3$, ...
- Traverse table diagonally from left-to-right, in the order shown by arrows

Enumerating the Rationals, cont.



- This allows us to list all rationals in a sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \ldots$$

- Hence, set of rationals is countable

Uncountability of Real Numbers

- Prime example of uncountably infinite sets is real numbers
- The fact that $\mathbb{R}$ is uncountably infinite was proven by George Cantor using the famous Cantor's diagonalization argument
- Reminiscient of Russell's paradox

Cantor's Diagonalization Argument

- For contradiction, assume the set of reals was countable
- Since any subset of a countable set is also countable, this would imply the set of reals between 0 and 1 is also countable
- Now, if reals between 0 and 1 are countable, we can list them in the following way:

$$\begin{array}{lllllll} R_1 = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\ R_2 = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\ R_3 = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\ \vdots & & & & & \ddots & \vdots & \\ R_n = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\ \vdots & & & & & & \vdots & \ddots \end{array}$$

Diagonalization Argument, cont

$$\begin{array}{lllllll} R_1 = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\ R_2 = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\ R_3 = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\ \vdots & & & & & \ddots & \vdots & \\ R_n = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\ \vdots & & & & & & \vdots & \ddots \end{array}$$

- Now, we'll create a new real number R and show that it is not equal to any of the R_i's in this sequence:
- Let $R = 0.a_1 a_2 a_3 \ldots$ such that:

$$a_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$$

- Clearly, this new number R differs from each number R_i in the table in at least one digit (its i'th digit)

Diagonalization Argument, concluded

- Since R is not in the table, this is not a complete enumeration of all reals between 0 and 1
- Hence, the set of real between 0 and 1 is not countable
- Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite

□