CS311H: Discrete Mathematics

Introduction to Graph Theory

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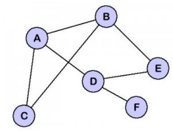
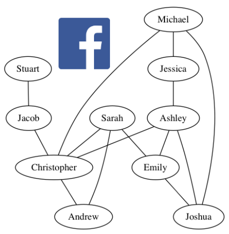
Motivation



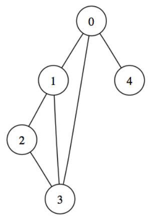
- Graph is a fundamental mathematical structure in computer science
- Graph $G = (V, E)$ consists of a set of vertices (nodes) V and edges E between these nodes
- Lots of applications in many areas: web search, transportation, biological models, ...
- Will encounter graphs and graph algorithms in many different courses

Example: Social Network as a Graph



- Nodes represent users (Michael, Jessica, Stuart ...)
- Edges represent friendship (e.g., Michael is friends with Jessica)
- Edge between nodes u and v is written as (u, v)
- e.g., (Sarah, Andrew) is an edge in this graph.

Terminology



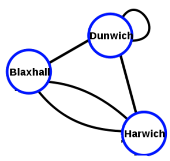
- Two nodes u and v are adjacent if there exists an edge between them (e.g., nodes 1 and 3)
- An edge (u, v) is incident with nodes u and v
- Degree of a vertex v, written $\deg(v)$, is the number of edges incident with it
- Neighborhood of a vertex is the set of vertices adjacent to it

Question

Consider a graph G with vertices v_1, v_2, v_3, v_4 and edges $(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_2, v_4)$.

1. Draw this graph.
2. What is the degree of each vertex?

Simple Graphs



- Graph contains a loop if any node is adjacent to itself
- A simple graph does not contain loops and there exists at most one edge between any pair of vertices
- Graphs that have multiple edges connecting two vertices are called multi-graphs
- Most graphs we will look at are simple graphs

Question

Consider a simple graph G where two vertices A and B have the same neighborhood.

Which of the following statements must be true about G?

A. The degree of each vertex must be even.

B. Both A and B have a degree of 0.

C. There cannot be an edge between A and B.

Handshaking Theorem

Let $G = (V, E)$ be a graph with m edges. Then:

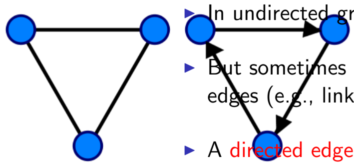
$$\sum_{v \in V} \deg(v) = 2m$$

- Intuition: Each edge contributes two to the sum of the degrees
- Proof: By induction on the number of edges.
- Base case:
- Induction:

Applications of Handshaking Theorem

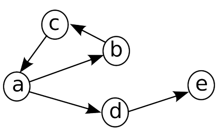
- Is it possible to construct a graph with 5 vertices where each vertex has degree 3?
- Prove that every graph has an even number of vertices of odd degree.
- If n people go to a party and everyone shakes everyone else's hand, how many handshakes occur?

Directed Graphs



- All graphs we considered so far are undirected
- In undirected graphs, edge (u, v) same as (v, u)
- But sometimes necessary to assign directions to edges (e.g., links from one webpage to another)
- A directed edge (arc) is an ordered pair (u, v) (i.e., (u, v) not same as (v, u))
- A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

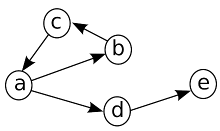


- The in-degree of a vertex v, written $\deg^-(v)$, is the number of edges going into v
- $\deg^-(a) =$
- The out-degree of a vertex v, written $\deg^+(v)$, is the number of edges leaving v
- $\deg^+(a) =$

Handshaking Theorem for Directed Graphs

Let $G = (V, E)$ be a directed graph. Then:

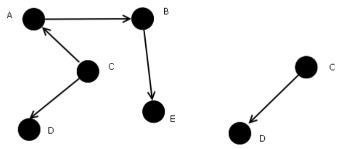
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$



- $\sum_{v \in V} \deg^-(v) =$
- $\sum_{v \in V} \deg^+(v) =$

Subgraphs

- A graph $G = (V, E)$ is a subgraph of another graph $G' = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$
- Example:



- Graph G is a proper subgraph of G' if $G \neq G'$.

Question

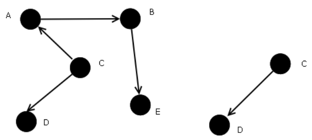
Consider a graph G with vertices $\{(v_1, v_2, v_3, v_4)\}$ and edges $(v_1, v_3), (v_1, v_4), (v_2, v_3)$.

Which of the following are subgraphs of G?

1. Graph G_1 with vertex v_1 and edge (v_1, v_3)
2. Graph G_2 with vertices $\{v_1, v_3\}$ and edge (v_1, v_3)
3. Graph G_3 with vertices $\{v_1, v_3\}$ and no edges
4. Graph G_4 with vertices $\{v_1, v_2\}$ and edge (v_1, v_2)

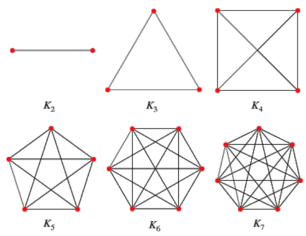
Induced Subgraph

- Consider a graph $G = (V, E)$ and a set of vertices V' such that $V' \subseteq V$
- Graph G' is the induced subgraph of G with respect to V' if:
  1. G' contains exactly those vertices in V'
  2. For all $u, v \in V'$, edge $(u, v) \in G'$ iff $(u, v) \in G$
- Subgraph induced by vertices $\{C, D\}$:



Complete Graphs

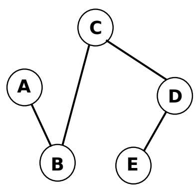
- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.



- How many edges does a complete graph with n vertices have?

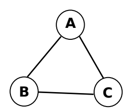
Bipartite graphs

- A simple undirected graph $G = (V, E)$ is called bipartite if V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in E connects a V_1 vertex to a V_2 vertex

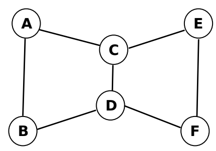


Examples Bipartite and Non-Bi-partite Graphs

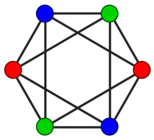
- Is this graph bipartite?



- What about this graph?



Graph Coloring



- A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is k-colorable if it is possible to color it using k colors.
  - e.g., graph on left is 3-colorable
  - Is it also 2-colorable?
- The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Question

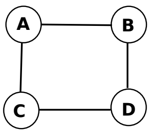
Consider a graph G with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)$.

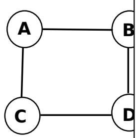
Which of the following are valid colorings for G?

1. v_1 = red, v_2 = green, v_3 = blue
2. v_1 = red, v_2= green, v_3 = blue, v_4 = red
3. v_1 = red, v_2= green, v_3 = red, v_4 = blue

Examples

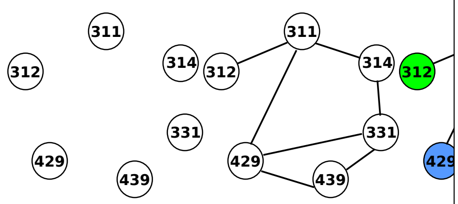
What are the chromatic numbers for these graphs?





Applications of Graph Coloring

- Graph coloring has lots of applications, particularly in scheduling.
- Example: What's the minimum number of time slots needed so that no student is enrolled in conflicting classes?



A Scheduling Problem

- The math department has 6 committees $C_1, \ldots, C_n$ that meet once a month.
- The committee members are:

$$C_1 = \{\text{Allen, Brooks, Marg}\} \quad C_2 = \{\text{Brooks, Jones, Morton}\}$$
$$C_3 = \{\text{Allen, Marg, Morton}\} \quad C_4 = \{\text{Jones, Marg, Morton}\}$$
$$C_5 = \{\text{Allen, Brooks}\} \quad C_6 = \{\text{Brooks, Marg, Morton}\}$$

- How many different meeting times must be used to guarantee that no one has conflicting meetings?

Bipartite Graphs and Colorability

Prove that a graph $G = (V, E)$ is bipartite if and only if it is 2-colorable.

Complete graphs and Colorability

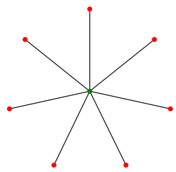
Prove that any complete graph K_n has chromatic number n.

Degree and Colorability

Theorem: Let G be a simple graph such that $\text{max\_degree}(G) = n$. Then, G is $n + 1$-colorable.

Degree and Colorability, cont.

Star Graphs and Colorability



- A star graph S_n is a graph with one vertex u at the center and the only edges are from u to each of $v_1, \ldots, v_{n-1}$.
- Draw S_2, S_3, S_4, S_5.
- What is the chromatic number of S_n?

Question About Star Graphs

Suppose we have two star graphs S_k and S_m. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph G?

1. The chromatic number of G is 3
2. G is 2-colorable.
3. $\text{max\_degree}(G) = \max(k, m)$.