CS311H: Discrete Mathematics

Graph Theory II

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Review

- What does it mean for a graph to be bipartite?
- What is the chromatic number of a graph?
- How do you prove that the chromatic number is n?

Bipartite Graphs and Colorability

Prove that a graph $G = (V, E)$ is bipartite if and only if it is 2-colorable.

Complete graphs and Colorability

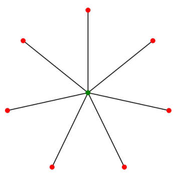
Prove that any complete graph K_n has chromatic number n.

Degree and Colorability

Theorem: Let G be a simple graph such that $\max\_degree(G) = n$. Then, G is $n + 1$-colorable.

Degree and Colorability, cont.

Star Graphs and Colorability



- A star graph S_n is a graph with one vertex u at the center and the only edges are from u to each of $v_1, \ldots, v_{n-1}$.
- Draw S_2, S_3, S_4, S_5.
- What is the chromatic number of S_n?

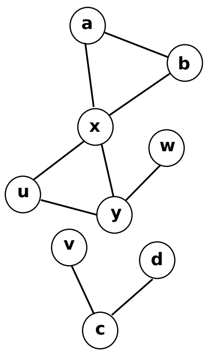
Question About Star Graphs

Suppose we have two star graphs S_k and S_m. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph G?

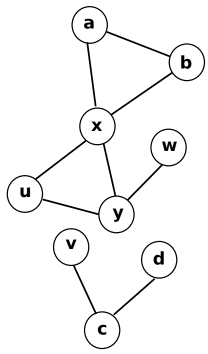
1. The chromatic number of G is 3
2. G is 2-colorable.
3. $\text{max\_degree}(G) = \max(k, m)$.

Connectivity in Graphs



- Typical question: Is it possible to get from some node u to another node v?
- Example: Train network – if there is path from u to v, possible to take train from u to v and vice versa.
- If it's possible to get from u to v, we say u and v are connected and there is a path between u and v

Paths

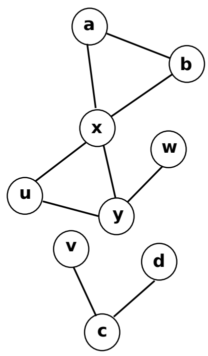


- A path between u and v is a sequence of edges that starts at vertex u, moves along adjacent edges, and ends in v.
- Example: u, x, y, w is a path, but u, y, v and u, a, x are not
- Length of a path is the number of edges traversed, e.g., length of u, x, y, w is 3
- A simple path is a path that does not repeat any edges
- u, x, y, w is a simple path but u, x, u is not

Example

- Consider a graph with vertices $\{x, y, z, w\}$ and edges $(x, y), (x, w), (x, z), (y, z)$
- What are all the simple paths from z to w?
- What are all the simple paths from x to y?
- How many paths (can be non-simple) are there from x to y?

Connectedness



- A graph is connected if there is a path between every pair of vertices in the graph
- Example: This graph not connected; e.g., no path from x to d
- A connected component of a graph G is a maximal connected subgraph of G

Example

- Prove: Suppose graph G has exactly two vertices of odd degree, say u and v. Then G contains a path from u to v.
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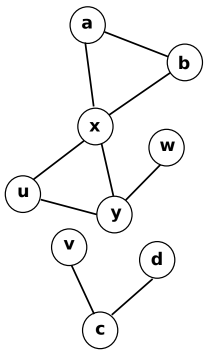
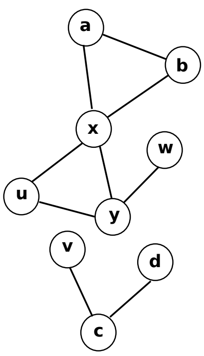
Circuits



- A circuit is a path that begins and ends in the same vertex.
- u, x, y, x, u and u, x, y, u are both circuits
- A simple circuit does not contain the same edge more than once
- u, x, y, u is a simple circuit, but u, x, y, x, u is not
- Length of a circuit is the number of edges it contains, e.g., length of u, x, y, u is 3
- In this class, we only consider circuits of length 3 or more

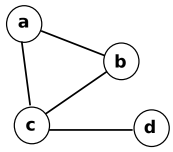
Cycles



- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, u, x, a, b, x, y, u is a circuit but not a cycle
- However, u, x, y, u is a cycle

Example

- Prove: If a graph has an odd length circuit, then it also has an odd length cycle.
- Huh? Recall that not every circuit is a a cycle.
- According to this theorem, if we can find an odd length circuit, we can also find odd length cycle.
- Example: d, c, a, b, c, d is an odd length circuit, but graph also contains odd length cycle



Proof

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Proof by strong induction on the length of the circuit.
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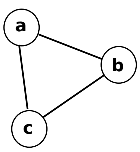

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Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

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Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

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Colorability and Cycles

Prove: If a graph is 2-colorable, then all cycles are of even length.

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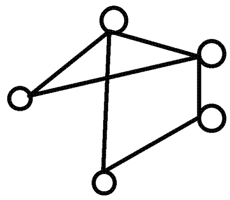
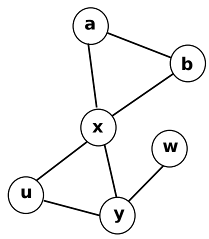
Example



- Is this graph 2-colorable?

Distance Between Vertices



- The distance between two vertices u and v is the length of the shortest path between u and v
- What is the distance between u and b?
- What is the distance between u and x?
- What is the distance between x and w?

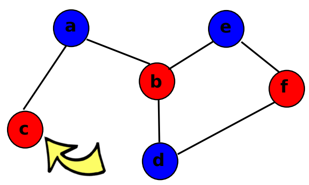
More Colorability and Cycles

Prove: If graph has no odd length cycles, then graph is 2-colorable.

- To prove this, we first consider an algorithm for coloring the graph with two colors.
- Then, we will show that this algorithm works if graph does not have odd length cycles.

The Algorithm

- Pick any vertex v in the graph.
- If a vertex u has odd distance from v, color it blue
- Otherwise, color it red

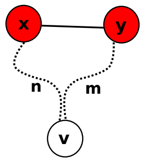


Proof

- We will now prove: "If the graph does not have odd length cycles, the algorithm is correct."
- Correctness of the algorithm implies graph is 2-colorable.
- Proof by contradiction.
- Suppose graph does not have odd length cycles, but the algorithm produces an invalid coloring.
- Means there exist two vertices x and y that are assigned the same color.

Proof, cont.

- Case 1: They are both assigned red



- We know n, m are both even
- This means we now have an odd-length circuit involving n, m
- By theorem from earlier, this implies that graph has odd length cycle, i.e., contradiction
- Case 2 is exactly the same.

Putting It All Together

- Theorem: A graph is 2-colorable if and only if it does not have odd-length cycles
- Corollary: A graph is bipartite if and only if it does not have odd-length cycles
- Example: Consider a graph G with vertices a, b, c, d, e, f
  - Is G partitle if its edges are $(a, f), (e, f), (e, d), (c, d), (a, c)$?