CS311H: Discrete Mathematics

Graph Theory III

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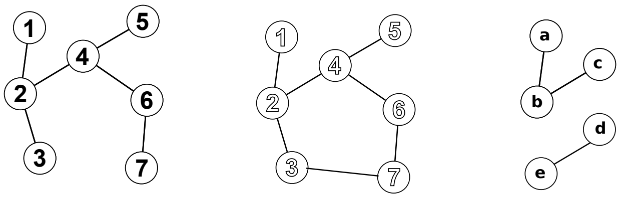
Trees

- A tree is a connected undirected graph with no cycles.
- Examples and non-examples:



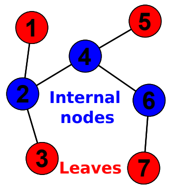
- An undirected graph with no cycles is a forest.

Fact About Trees

Theorem: An undirected graph G is a tree if and only if there is a unique simple path between any two of its vertices.

Leaves of a Tree

- Given a tree, a vertex of degree 1 is called a leaf.



- **Important fact: Every tree with more than 2 nodes has at least two leaves.**

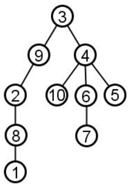
Why is this true?

Number of Edges in a Tree

Theorem: A tree with n vertices has $n-1$ edges.

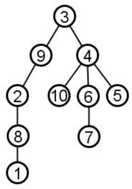
- Proof is by induction on n
- Base case: $n = 1$ ✓
- Induction: Assume property for tree with n vertices, and show tree T with $n+1$ verices has n edges
- Construct T' by removing a leaf from T; T' is a tree with n vertices (tree because connected and no cycles)
- By IH, T' has $n-1$ edges
- Add leaf back: $n+1$ vertices and n edges

Rooted Trees



- A rooted tree has a designated root vertex and every edge is directed away from the root.
- Vertex v is a parent of vertex u if there is an edge from v to u; and u is called a child of v
- Vertices with the same parent are called siblings
- Vertex v is an ancestor of u if v is u's parent or an ancestor of u's parent.
- Vertex v is a descendant of u if u is v's ancestor

Questions about Rooted Trees

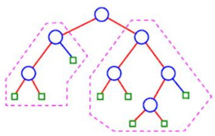


- Suppose that vertices u and v are siblings in a rooted tree.
- Which statements about u and v are true?

1. They must have the same ancestors
2. They can have a common descendant
3. If u is a leaf, then v must also be a leaf

Subtrees

- Given a rooted tree and a node v, the subtree rooted at v includes v and its descendants.



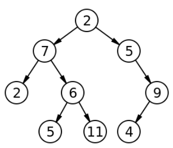
- Level of vertex v is the length of the path from the root to v.
- The height of a tree is the maximum level of its vertices.

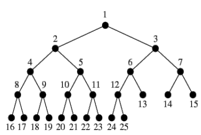
True-False Questions

1. Two siblings u and v must be at the same level.
2. A leaf vertex does hot have a subtree.
3. The subtrees rooted at u and v can have the same height only if u and v are siblings.
4. The level of the root vertex is 1.

m-ary Trees

- A rooted tree is called an m-ary tree if every vertex has no more than m children.
- An m-ary tree where $m = 2$ is called a binary tree.
- A full m-ary tree is a tree where every internal node has exactly m children.
- Which are full binary trees?





Useful Theorem

Theorem: An m-ary tree of height $h \geq 1$ contains at most m^h leaves.

- Proof is by induction on height h.
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Corollary

Corollary: If m-ary tree has height h and n leaves, then $h \geq \lceil log_m n \rceil$

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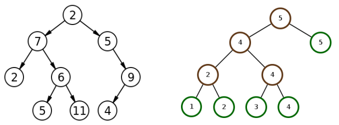
Questions

- What is maximum number of leaves in binary tree of height 5?
- If binary tree has 100 leaves, what is a lower bound on its height?
- If binary tree has 2 leaves, what is an upper bound on its height?

Balanced Trees

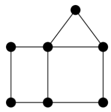
- An m-ary tree is balanced if all leaves are at levels h or $h-1$



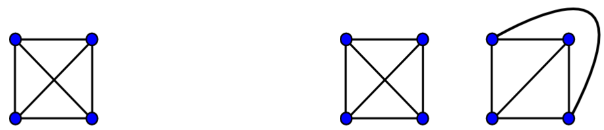
- "Every full tree must be balanced." – true or false?
- "Every balanced tree must be full." – true or false?

Planar Graphs

- A graph is called planar if it can be drawn in the plane without any edges crossing (called planar representation).



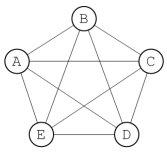
- Is this graph planar?



- In this class, we will assume that every planar graph has at least 3 edges.

A Non-planar Graph

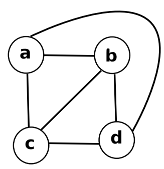
- The complete graph K_5 is not planar:



- Why can K_5 not be drawn without any edges crossing?

Regions of a Planar Graph

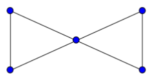
- The planar representation of a graph splits the plane into regions (sometimes also called faces):



- Degree of a region R, written $\deg(R)$, is the number of edges bordering R
- Here, all regions have degree 3.

Examples

- How many regions does this graph have?

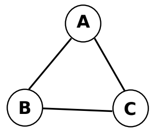


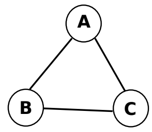
- What is the degree of its outer region?
- How many regions does a graph have if it has no cycles?
- Given a planar simple graph with at least 3 edges, what is the minimum degree a region can have?
- What is the relationship between $\sum \deg(R)$ and the number of edges?

Euler's Formula

Euler's Formula: Let $G = (V, E)$ be a planar connected graph with regions R. Then, the following formula always holds:

$$|R| = |E| - |V| + 2$$





All planar representations of a graph split the plane into the same number of regions!

Proof of Euler's Formula

- Case 1: G does not have cycles (i.e., a tree)
- If G has $|V|$ nodes, how many edges does it have?
- How many regions does it have?
- $|R| = 1 = (|V| - 1) - |V| + 2$ ✓

Proof, cont.

- Case 2: G has at least one cycle.
- The proof is by induction on the number of edges.
- Base case: G has 3 edges (i.e., a triangle)
- Induction: Suppose Euler's formula holds for planar connected graphs with e edges and at least one cycle.
- We need to show it also holds for planar connected graphs with $e + 1$ edges and at least one cycle.

Proof, cont.

- Create G' by removing one edge from the cycle $\Rightarrow$ has e edges
- If G' doesn't have cycles, we know $|R| = e - |V| + 2$ (case 1)
- If G' has cycles, we know from IH that $|R| = e - |V| + 2$
- Now, add edge back in; G has $e + 1$ edges and $|V|$ vertices
- How many regions does G have? $|R| + 1$
- $e + 1 - |V| + 2 = |R| + 1$ ✓

An Application of Euler's Formula

- Suppose a connected planar simple graph G has 6 vertices, each with degree 4.
- How many regions does a planar representation of G have?
- How many edges?
- How many regions?

A Corollary of Euler's Formula

Theorem: Let G be a connected planar simple graph with v vertices and e edges. Then $e \leq 3v - 6$

- Proof: Suppose G has r regions.
- Recall: $2e = \sum \deg(R)$
- Hence, $2e \geq 3r$
- From Euler's formula, $3r = 3e - 3v + 6$; thus $2e \geq 3e - 3v + 6$
- Implies $e \leq 3v - 6$ ✓

Why is this Theorem Useful?

Theorem: Let G be a connected planar simple graph with v vertices and e edges. Then $e \leq 3v - 6$

- Can be used to show graph is not planar.
- Example: Prove that K_5 is not planar.
- How many edges does K_5 have?
- $3 \cdot 5 - 6 = 9$, but $10 \not\leq 9$

Another Corollary

Theorem: If G is a connected, planar simple graph, then it has a vertex of degree not exceeding 5.

- Proof by contradiction: Suppose every vertex had degree at least 6
- What lower bound does this imply on number of edges?
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