CS311H: Discrete Mathematics

Propositional Logic II

Instructor: Işıl Dillig

Converse of a Implication

- Recall implication $p \to q$ – when does it evaluate to false?
- The converse of an implication $p \to q$ is $q \to p$.
- What is the converse of "If I am a CS major, then I can program"?
- What is the converse of "If I get an A in CS311, then I am smart"?
- Is it possible for a implication to be true, but its converse to be false?

Inverse of an Implication

- The inverse of an implication $p \to q$ is $\neg p \to \neg q$.
- What is the inverse of "If I am a CS major, then I can program"?
- What is the inverse of "If I get an A in CS311, then I am smart"?
- Is it possible for a implication to be true, but its inverse to be false?

Contrapositive of Implication

- The contrapositive of an implication $p \to q$ is $\neg q \to \neg p$.
- What is the contrapositive of "If I am a CS major, then I can program"?
- What is the contrapositive of "If I get an A in CS311, then I am smart"?
- Question: Is it possible for an implication to be true, but its contrapositive to be false?

Conditional and its Contrapositive

A conditional $p \to q$ and its contrapositive $\neg q \to \neg p$ always have the same truth value.

- Proof: We consider all four possible cases:
  - $p = T, q = T$: Both $T \to T$ and $F \to F$ are true
  - $p = T, q = F$: Both $T \to F$ and $T \to F$ are false
  - $p = F, q = T$: Both $F \to T$ and $F \to T$ are true
  - $p = F, q = F$: Both $F \to F$ and $T \to T$ are true

Question

- Consider a conditional $p \to q$
- Is it possible that its converse is true, but inverse is false?

Summary

- Conditional is of the form $p \to q$
- Converse: $q \to p$
- Inverse: $\neg p \to \neg q$
- Contrapositive: $\neg q \to \neg p$
- Conditional and contrapositive have same truth value
- Inverse and converse always have same truth value

Biconditionals

- A biconditional $p \leftrightarrow q$ is the proposition "p if and only if q".
- The biconditional $p \leftrightarrow q$ is true if p and q have same truth value, and false otherwise.
- Exercise: Construct a truth table for $p \leftrightarrow q$
- Question: How can we express $p \leftrightarrow q$ using the other boolean connectives?

Operator Precedence

- Given a formula $p \land q \lor r$, do we parse this as $(p \land q) \lor r$ or $p \land (q \lor r)$?
- Without settling on a convention, formulas without explicit paranthesization are ambiguous.
- To avoid ambiguity, we will specify precedence for logical connectives.

Operator Precedence, cont.

- Negation ($\neg$) has higher precedence than all other connectives.
- Question: Does $\neg p \land q$ mean (i) $\neg(p \land q)$ or (ii) $(\neg p) \land q$?
- Conjunction ($\land$) has next highest predence.
- Question: Does $p \land q \lor q$ mean (i) $(p \land q) \lor r$ or (ii) $p \land (q \lor r)$?
- Disjunction ($\lor$) has third highest precedence.
- Next highest is precedence is $\to$, and lowest precedence is $\leftrightarrow$

Operator Precedence Example

- Which is the correct interpretation of the formula

$$p \lor q \land r \leftrightarrow q \to \neg r$$

(A) $((p \lor (q \land r)) \leftrightarrow q) \to (\neg r)$

(B) $((p \lor q) \land r) \leftrightarrow q) \to (\neg r)$

(C) $(p \lor (q \land r)) \leftrightarrow (q \to (\neg r))$

(D) $(p \lor ((q \land r) \leftrightarrow q)) \to (\neg r)$

Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- Example: $\neg p$ evaluates to true under under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for every assignment, e.g., $p \lor \neg p$
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., $p \land \neg p$
- Such formulas are called unsatisfiable formulas or contradictions

Interpretations

- To make satisfiability/validity precise, we'll define interpretation of formula
- An interpretation I for a formula F is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \cdots\}$$

- Each interpretation corresponds to one row in the truth table, so 2^n possible interpretations

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

Formula F + Interpretation I = Truth value

- We write $I \models F$ if F evaluates to true under I
- Similarly, $I \not\models F$ if F evaluates to false under I.
- Theorem: $I \models F$ if and only if $I \not\models \neg F$

Examples

- Consider the formula $F: \ p \land q \rightarrow \neg p \lor \neg q$
- Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{false}]$
- What does F evaluate to under I_1?
- Thus, $I_1 \models F$
- Let I_2 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$
- What does F evaluate to under I_2?
- Thus, $I_2 \not\models F$

Another Example

- Let F_1 and F_2 be two propositional formulas
- Suppose F_1 evaluates to true under interpretation I
- What does $F_2 \land \neg F_1$ evaluate to under I?

Satisfiability, Validity

- F is satisfiable iff there exists interpretation I s.t. $I \models F$
- F is valid iff for all interpretations I, $I \models F$
- F is unsatisfiable iff for all interpretations I, $I \not\models F$
- F is contingent if it is satisfiable, but not valid.

True/False Questions

Are the following statements true or false?

- If a formula is valid, then it is also satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable.
- If F_1 and F_2 are satisfiable, then $F_1 \land F_2$ is also satisfiable.
- If F_1 and F_2 are satisfiable, then $F_1 \lor F_2$ is also satisfiable.

Duality Between Validity and Unsatisfiability

F is valid if and only if $\neg F$ is unsatisfiable

- Proof:

Proving Validity

- Question: How can we prove that a propositional formula is a tautology?
- Exercise: Which formulas are tautologies? Prove your answer.
  1. $(p \to q) \leftrightarrow (\neg q \to \neg p)$
  2. $(p \land q) \lor \neg p$

Proving Satisfiability, Unsatisfiability, Contingency

- Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:
  - Satisfiable: There exists a row where formula evaluates to true
  - Unsatisfiable: In all rows, formula evaluates to false
  - Contingent: Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercise

- Is $(p \to q) \to (q \to p)$ valid, unsatisfiable, or contingent? Prove your answer.

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \to F_2$

$F_1 \Rightarrow F_2$ iff $F_1 \to F_2$ is valid

- Caveat: $F_1 \Rightarrow F_2$ is not a propositional logic formula; $\Rightarrow$ is not part of PL syntax!
- Instead, $F_1 \Rightarrow F_2$ is a semantic judgment, like satisfiability!

Example

- Does $p \lor q$ imply p? Prove your answer.

Equivalence

- Two formulas F_1 and F_2 are equivalent if they have same truth value for every interpretation, e.g., $p \lor p$ and p
- More precisely, formulas F_1 and F_2 are equivalent, written $F_1 \equiv F_2$ or $F_1 \Leftrightarrow F_2$, iff:

  $F_1 \Leftrightarrow F_2$ iff $F_1 \leftrightarrow F_2$ is valid

- $\equiv, \Leftrightarrow$ not part of PL syntax; they are semantic judgments!

Example

- Prove that $p \to q$ and $\neg p \lor q$ are equivalent

Important Equivalences

- Some important equivalences are useful to know!
- Law of double negation: $\neg\neg\phi \equiv \phi$
- Identity Laws: $\phi \land T \equiv \phi \qquad \phi \lor F \equiv \phi$
- Domination Laws: $\phi \lor T \equiv T \qquad \phi \land F \equiv F$
- Idempotent Laws: $\phi \lor \phi \equiv \phi \qquad \phi \land \phi \equiv \phi$
- Negation Laws: $\phi \land \neg\phi \equiv F \quad \phi \lor \neg\phi \equiv T$
- Absorption Laws: $\phi_1 \land (\phi_1 \lor \phi_2) \equiv \phi_1 \quad \phi_1 \lor (\phi_1 \land \phi_2) = \phi_2$

Commutativity and Distributivity Laws

- Commutative Laws: $\phi_1 \lor \phi_2 \equiv \phi_2 \lor \phi_1 \qquad \phi_1 \land \phi_2 \equiv \phi_2 \land \phi_1$
- Distributivity Law #1:
  $(\phi_1 \lor (\phi_2 \land \phi_3)) \equiv (\phi_1 \lor \phi_2) \land (\phi_1 \lor \phi_3)$
- Distributivity Law #2:
  $(\phi_1 \land (\phi_2 \lor \phi_3)) \equiv (\phi_1 \land \phi_2) \lor (\phi_1 \land \phi_3)$
- Associativity Laws: $\phi_1 \lor (\phi_2 \lor \phi_3) \equiv (\phi_1 \lor \phi_2) \lor \phi_3$
  $\phi_1 \land (\phi_2 \land \phi_3) \equiv (\phi_1 \land \phi_2) \land \phi_3$

De Morgan's Laws

- Let cs311 be the proposition "John took CS311" and cs314 be the proposition "John took CS314"
- In simple English what does $\neg(cs311 \land cs314)$ mean?
- DeMorgan's law expresses exactly this equivalence!
- De Morgan's Law #1: $\neg(p \land q) \equiv (\neg p \lor \neg q)$
- De Morgan's Law #2: $\neg(p \lor q) \equiv (\neg p \land \neg q)$
- When you "push" negations in, $\land$ becomes $\lor$ and vice versa

Why are These Equivalences Useful?

- Use known equivalences to prove that two formulas are equivalent
- i.e., rewrite one formula into another using known equivalences
- Examples: Prove following formulas are equivalent:
  1. $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$
  2. $\neg(p \to q)$ and $p \land \neg q$

Formalizing English Arguments in Logic

- We can use logic to prove correctness of English arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let f be the proposition "Joe drives fast", and t be the proposition "Joe gets a ticket"
- How do we encode this argument as a logical formula?

Example, cont

"If Joe drives fast, he gets a speeding ticket. Joe did not get a ticket. Therefore, he did not drive fast.": $((f \rightarrow t) \land \neg t) \rightarrow \neg f$

- How can we prove this argument is valid?
- Can do this in two ways:
  1. Use truth table to show formula is tautology
  2. Use known equivalences to rewrite formula to true

Another Example

- Can also use to logic to prove an argument is not valid.
- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Let's use logic to prove George's argument doesn't hold water.
- Let u = "Jill is carrying an umbrella", and r = "It is raining"
- How do we encode this argument in logic?

Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining.": $((u \rightarrow r) \land \neg u) \rightarrow \neg r$

- How can we prove George's argument is invalid?

Summary

- A formula is valid if it is true for all interpretations.
- A formula is satisfiable if it is true for at least one interpretation.
- A formula is unsatisfiable if it is false for all interpretations.
- A formula is contingent if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas F_1 and F_2 are equivalent, written $F_1 \equiv F_2$, if $F_1 \leftrightarrow F_2$ is valid