CS311H: Discrete Mathematics

Graph Theory IV

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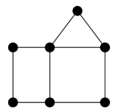
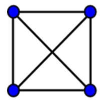
Planar Graphs

- A graph is called planar if it can be drawn in the plane without any edges crossing (called planar representation).



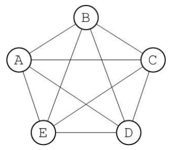
- Is this graph planar?



- In this class, we will assume that every planar graph has at least 3 edges.

A Non-planar Graph

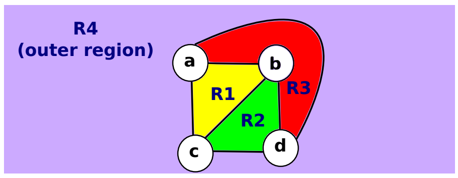
- The complete graph K_5 is not planar:



- Why can K_5 not be drawn without any edges crossing?

Regions of a Planar Graph

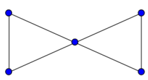
- The planar representation of a graph splits the plane into regions (sometimes also called faces):



- Every planar graph has an outer region, which is unbounded.
- Degree of a region R, written $\deg(R)$, is the number of edges adjacent to R
- What is degree of $R1, R2, R3, R4$?

Examples

- How many regions does this graph have?

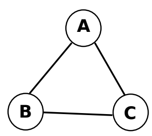


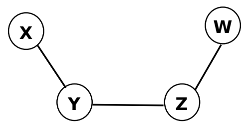
- What is the degree of its outer region?
- How many regions does a graph have if it has no cycles?

Euler's Formula

Euler's Formula: Let $G = (V, E)$ be a planar connected graph with regions R. Then, the following formula always holds:

$$|R| = |E| - |V| + 2$$





All planar representations of a graph split the plane into the same number of regions!

Proof of Euler's Formula

- Case 1: G does not have cycles (i.e., a tree)
- If G has $|V|$ nodes, how many edges does it have?
- How many regions does it have?
- $|R| = 1 = (|V| - 1) - |V| + 2 \quad \checkmark$

Proof, cont.

- Case 2: G has at least one cycle.
- The proof is by induction on the number of edges.
- Base case: G has 3 edges (i.e., a triangle)
- Induction: Suppose Euler's formula holds for planar connected graphs with e edges and at least one cycle.
- We need to show it also holds for planar connected graphs with $e + 1$ edges and at least one cycle.

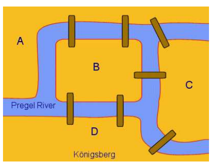
Proof, cont.

- Create G' by removing one edge from the cycle $\Rightarrow$ has e edges
- If G' doesn't have cycles, we know $|R| = e - |V| + 2$ (case 1)
- If G' has cycles, we know from IH that $|R| = e - |V| + 2$
- Now, add edge back in; G has $e + 1$ edges and $|V|$ vertices
- How many regions does G have? $|R| + 1$
- $e + 1 - |V| + 2 = |R| + 1 \quad \checkmark$

An Application of Euler's Formula

- Suppose a connected planar simple graph G has 6 vertices, each with degree 4.
- How many regions does a planar representation of G have?
- How many edges?
- How many regions?

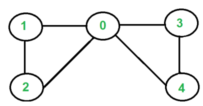
Seven Bridges of Königsberg



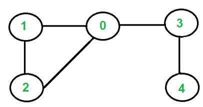
- Town of Königsberg in Germany divided into four parts by the Pregel river and had seven bridges
- Townspeople wondered if one can start at point A, cross all bridges exactly once, and come back to A
- Mathematician Euler heard about this puzzle and solved it

Euler Circuits and Euler Paths

- Given graph G, an Euler circuit is a simple circuit containing every edge of G.

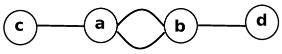


- Euler path is a simple path containing every edge of G.

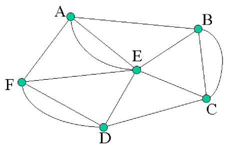


Examples

- Does this graph have a Euler circuit or path?



- What about this one?



- Are there some criteria that allow us to easily determine if a graph has Euler circuit or path?

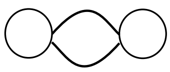
Theorem about Euler Circuits

Theorem: A connected multigraph G with at least two vertices contains an Euler circuit if and only if each vertex has even degree.

- Let's first prove the "only if" part.
- Euler circuit must enter and leave each vertex the same number of times.
- But we can't use any edge twice
- Hence, each vertex must have even number of adjacent edges.

Proof of Sufficiency

- Now, prove the if part – much more difficult!
- By strong induction on the number of edges e
- Base case: $e = 2$



- Induction: Suppose claim holds for every graph with $\leq e$ edges; show it holds for graph with $e + 1$ edges
- Consider graph G with $e + 1$ edges and where every vertex has even degree
- Observe: G cannot be a tree – why?

Proof, cont.

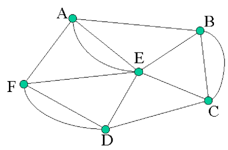
- This means G must contain a cycle, say C
- Now, remove all edges in C from G to obtain graph G'
- G' may not be connected, suppose it consists of connected components $G_1, \ldots, G_n$
- Each vertex in a cycle has exactly two adjacent edges that are part of the cycle
- Hence, if all nodes in G have even degree, then nodes in each G_i must also have even degree

Proof, cont.

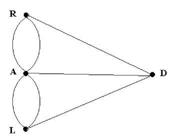
- Now, each G_i is connected and every vertex has even degree
- By IH, each G_i has an Euler circuit, say C_i
- We can now also build an Euler circuit for G using these C_i's
- Start at some vertex v in C, traverse along C until we reach a vertex v_i in connected component G_i
- Now, traverse C_i and come back to v_i
- Continue until we are back at v_i
- This is an Euler circuit because we've traversed every edge and haven't repeated any edges

Revisiting Example

- Does this graph have an Euler circuit?



- An Euler circuit:
- Does this have an Euler circuit?



Necessary and Sufficient Conditions for Euler Paths

Theorem: A connected multigraph G contains an Euler path iff there are exactly 0 or 2 vertices of odd degree.

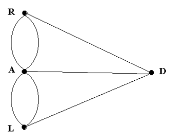
- Let's first prove necessity: Suppose G has Euler path P with start and end-points u and v
- Case 1: u, v are the same – then P is an Euler circuit, hence it must have 0 vertices of degree
- Case 2: u, v are distinct
- Except for u, v, we must enter and leave each vertex same number of times – these must have even degree
- We must leave u one more time than we enter it, and we enter v one more time than we leave it, so they have odd degree

Proof of Sufficiency

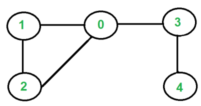
- Suppose G has exactly 0 or 2 vertices with odd degree
- Case 1: If no vertices with odd degree, must have Euler circuit
- Case 2: It has exactly two vertices, say u, v, with odd degree
- Now, add an edge between u, v to generate graph G'
- All vertices in G' have even degree – so G' has Euler circuit
- This means G has Euler path with start and end-points u, v

Example

- Does this graph have Euler path?

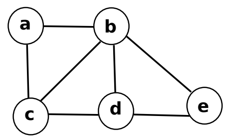


- Graph with an Euler path:

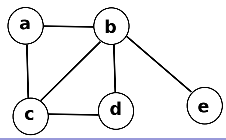


Hamilton Paths and Circuits

- A Hamilton circuit in a graph G is a simple circuit that visits every vertex in G exactly once (except the start node).

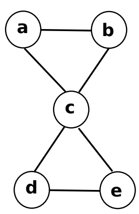


- Note that all Hamilton circuits are cycles!
- A Hamilton path in a graph G is a simple path that visits every vertex in G exactly once.



Are All Euler Circuits Also Hamilton Circuits?

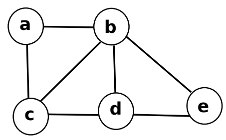
- Not every Euler circuit is a Hamilton circuit:



- Does this graph have a Hamilton path?

Hamilton vs Euler Circuits

- Not every Hamilton circuit is an Euler circuit:



- Find a graph that has an Euler circuit, but no Hamilton circuit

Necessary and Sufficient Criteria for Hamilton Circuits

- Unlike Euler circuits, no necessary and sufficient criteria for identifying Hamilton circuits or paths
- Exercise: Prove that a graph with a vertex of degree 1 cannot have a Hamilton circuit.
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