A Non-planar Graph

- The complete graph $K_5$ is not planar:

- Why can $K_5$ not be drawn without any edges crossing?

Regions of a Planar Graph

- The planar representation of a graph splits the plane into regions (sometimes also called faces):

- Every planar graph has an outer region, which is unbounded.
- Degree of a region $R$, written $\deg(R)$, is the number of edges adjacent to $R$
- What is degree of $R_1$, $R_2$, $R_3$, $R_4$?

Examples

- How many regions does this graph have?
- What is the degree of its outer region?
- How many regions does a graph have if it has no cycles?

Euler’s Formula

**Euler’s Formula**: Let $G = (V, E)$ be a planar connected graph with regions $R$. Then, the following formula always holds:

$$|R| = |E| - |V| + 2$$

All planar representations of a graph split the plane into the same number of regions!
Proof of Euler’s Formula

- **Case 1:** $G$ does not have cycles (i.e., a tree)
  - If $G$ has $|V|$ nodes, how many edges does it have?
  - How many regions does it have?
  - $|R| = 1 = (|V| - 1) - |V| + 2 \quad \checkmark$

Proof, cont.

- **Case 2:** $G$ has at least one cycle.
  - The proof is by induction on the number of edges.
    - **Base case:** $G$ has 3 edges (i.e., a triangle)
    - **Induction:** Suppose Euler’s formula holds for planar connected graphs with $e$ edges and at least one cycle.
      - We need to show it also holds for planar connected graphs with $e + 1$ edges and at least one cycle.

Create $G'$ by removing one edge from the cycle $\Rightarrow$ has $e$ edges

- If $G'$ doesn’t have cycles, we know $|R| = e - |V| + 2$ (case 1)
- If $G'$ has cycles, we know from IH that $|R| = e - |V| + 2$
- Now, add edge back in; $G$ has $e + 1$ edges and $|V|$ vertices
- How many regions does $G$ have? $|R| + 1$
- $e + 1 - |V| + 2 = |R| + 1 \quad \checkmark$

An Application of Euler’s Formula

- Suppose a connected planar simple graph $G$ has 6 vertices, each with degree 4.
  - How many regions does a planar representation of $G$ have?
  - How many edges?
  - How many regions?

Seven Bridges of Königsberg

- Town of Königsberg in Germany divided into four parts by the Pregel river and had seven bridges
- Townspeople wondered if one can start at point $A$, cross all bridges exactly once, and come back to $A$
- Mathematician Euler heard about this puzzle and solved it

Euler Circuits and Euler Paths

- Given graph $G$, an **Euler circuit** is a simple circuit containing every edge of $G$.

- **Euler path** is a simple path containing every edge of $G$. 

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Theorem about Euler Circuits

Theorem: A connected multigraph $G$ with at least two vertices contains an Euler circuit if and only if each vertex has even degree.

◮ Let’s first prove the “only if” part.

◮ Euler circuit must enter and leave each vertex the same number of times.

◮ But we can’t use any edge twice

◮ Hence, each vertex must have even number of adjacent edges.

Proof of Sufficiency

◮ Now, prove the if part – much more difficult!

◮ By strong induction on the number of edges $e$

◮ Base case: $e = 2$

◮ Induction: Suppose claim holds for every graph with $\leq e$ edges; show it holds for graph with $e + 1$ edges

◮ Consider graph $G$ with $e + 1$ edges and where every vertex has even degree

◮ Observe: $G$ cannot be a tree – why?

Proof, cont.

◮ Now, each $G_i$ is connected and every vertex has even degree

◮ By IH, each $G_i$ has an Euler circuit, say $C_i$

◮ We can now also build an Euler circuit for $G$ using these $C_i$’s

◮ Start at some vertex $v$ in $C$, traverse along $C$ until we reach a vertex $v_i$ in connected component $G_i$

◮ Now, traverse $C_i$ and come back to $v_i$

◮ Continue until we are back at $v_i$

◮ This is an Euler circuit because we’ve traversed every edge and haven’t repeated any edges

Revisiting Example

◮ Does this graph have an Euler circuit?

◮ An Euler circuit:

◮ Does this have an Euler circuit?