CS311H Problem Set 5

Due Thursday, October 25

1. (10 points) Prove by induction that the sum of the first \( n \) positive odd integers is \( n^2 \). Explicitly state whether you are using regular or strong induction.

2. (10 points) Use induction to prove that the following equality holds for every positive integer \( n \):

\[
\sum_{i=1}^{n} i \cdot 2^i = (n - 1) \cdot 2^{n+1} + 2
\]

State whether you are using regular or strong induction.

3. (5 points) Explain what is wrong with the following "proof":

Claim: \( \forall n \geq 0. \ 3n = 0 \)

Proof: By strong induction on \( n \). For the base case, we have \( n = 0 \). Since \( 3 \cdot 0 = 0 \), the claim holds for the base case.
For the inductive step, we prove the property for \( k + 1 \), i.e., \( 3 \cdot (k + 1) = 0 \). Observe that \( k + 1 \) can be written as \( i + j \) for some natural numbers \( i, j \) where \( i \leq k \) and \( j \leq k \). Now, by the inductive hypothesis, we have \( 3i = 0 \) and \( 3j = 0 \). Hence, \( 3 \cdot (k + 1) = 3i + 3j = 0 + 0 = 0 \). Hence the property holds.

4. (10 points) Prove by induction that \( 3^n < n! \) for all integers \( n \) greater than 6. State whether you are using regular or strong induction.

5. (15 points) Prove by induction that every integer \( n > 17 \) can be written in the form \( n = 4a + 7b \) where \( a, b \geq 0 \). State whether you are using regular or strong induction.

6. (10 points, 5 points each) Give a recursive definition of the following:
7. (15 points) Consider the subset $S$ of the set of ordered pairs of integers defined recursively as follows:

- Base case: $(0, 0) \in S$
- Recursive case: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$, and $(a + 3, b + 2) \in S$

Based on this definition:

(a) (5 points) List the first five elements in $S$

(b) (10 points) Use structural induction to show that $5 \mid (a + b)$ for all $(a, b) \in S$

8. (15 points) A bitstring is a string consisting of only 0’s and 1’s. Consider the following recursive definition of the function “count”, which counts the number of 1’s in the bitstring:

- Base case: $\text{count}(\epsilon) = 0$
- Recursive case 1: $\text{count}(1 \cdot s) = 1 + \text{count}(s)$
- Recursive case 2: $\text{count}(0 \cdot s) = \text{count}(s)$

Use structural induction to prove that $\text{count}(st) = \text{count}(s) + \text{count}(t)$. 

(a) the set of positive integers congruent to 4 modulo 5
(b) the sequence defined by $a_n = n(n + 1)$ for $n \geq 1$