1. (10 points) Let $G = (V, E)$ be a simple, undirected graph such that $\forall v \in V. \text{degree}(v) \geq 2$. Is the statement “$G$ must contain a cycle” true or false? If true, prove your answer. Otherwise, give a counterexample.

2. (10 points) Consider a graph $T'$ that is obtained by adding an arbitrary edge to some tree $T$. Can $T'$ be a tree? If yes, give an example of such a $T$ and $T'$. If not, prove that $T'$ can never be a tree.

3. (10 points) Consider a tree $T$ with maximum degree $m \geq 2$. Prove that $T$ has at least $m$ leaves.

4. (10 points) Let $G$ be a planar graph with $e$ edges such that $e \geq 3$ and suppose $G$ has at least 2 regions. Recall the Region Handshaking Theorem we discussed in class, which states:

$$\sum_{R \in \text{region}(G)} \text{deg}(R) = 2e$$

Prove the region handshaking theorem using induction.

5. (20 points) For each of the properties listed below, draw a connected multi-graph with at least three vertices and at least three edges that has these properties or explain why no such graph can exist. For each graph you draw, also explain why it has these properties.

(a) A bipartite graph with an odd number of vertices and contains an Euler circuit
(b) A bipartite graph with an odd number of vertices that also contains a Hamilton circuit

(c) A graph with three vertices that does not contain an Euler circuit, but contains an Euler path and a Hamilton circuit

(d) A bipartite graph that has a Hamilton circuit, but no Euler circuit