CS311H: Discrete Mathematics
Permutations and Combinations
Instructor: Isil Dillig

Announcements

- Homework 5 is due today
- Homework 6 is out today; due next Thursday
- Can leave answer in the form $2^{12}$ – don’t have to calculate exact answer

The Pigeonhole Principle

- Suppose there is a flock of 36 pigeons and a set of 35 pigeonholes
- Each pigeon wants to sit in one hole
- But since there are less holes than there are pigeons, one pigeon is left without a hole.

The Pigeonhole Principle: If $n + 1$ or more objects are placed into $n$ boxes, then at least one box contains 2 or more objects

Examples

- Consider an event with 367 people. Is it possible no pair of people have the same birthday?
- Consider function $f$ from a set with $k + 1$ or more elements to a set with $k$ elements. Is it possible $f$ is one-to-one?
- Consider $n$ married couples. How many of the $2n$ people must be selected to guarantee there is at least one married couple?

Generalized Pigeonhole Principle

- If $n$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil n/k \rceil$ objects

Proof: (by contradiction) Suppose every box contains less than $\lceil n/k \rceil$ objects

Examples

- If there are 30 students in a class, at least how many must be born in the same month?
- What is the minimum $\#$ of students required to ensure at least 6 students receive the same grade (A, B, C, D, F)?
- What is the min $\#$ of cards that must be chosen to guarantee three have same suit?
Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - No object can be selected more than once.
  - Order of arrangement matters.

Example: $S = \{a, b, c\}$. What are the permutations of $S$?

How Many Permutations?

- Consider set $S = \{a_1, a_2, \ldots, a_n\}$.
- How many permutations of $S$ are there?
- Decompose using product rule:
  - How many ways to choose first element?
  - How many ways to choose second element?
  - \ldots
  - How many ways to choose last element?
- What is number of permutations of set $S$?

Examples

- Consider the set $\{7, 10, 23, 4\}$. How many permutations?
- How many permutations of letters A, B, C, D, E, F, G contain "ABC" as a substring?
- \ldots

Computing $P(n, r)$

- Given a set with $n$ elements, what is $P(n, r)$?
- Decompose using product rule:
  - How many ways to pick first element?
  - How many ways to pick second element?
  - \ldots
  - How many ways to pick $r$th element?
  - How many ways to pick last element?
- Thus, $P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$

Examples

- What is the number of 2-permutations of set $\{a, b, c, d, e\}$?
- \ldots
- Salesman must visit 4 cities from list of 10 cities: Must begin in Chicago, but can choose the remaining cities and order.
  - How many possible itinerary choices are there?

$r$-Permutations

- $r$-permutation is ordered arrangement of $r$ elements in a set $S$.
  - $S$ can contain more than $r$ elements.
  - But we want arrangement containing $r$ of the elements in $S$.
  - The number of $r$-permutations in a set with $n$ elements is written $P(n, r)$.
- Example: What is $P(n, n)$?
Combinations

An r-combination of set S is the unordered selection of r elements from that set.

Unlike permutations, order does not matter in combinations.

Example: What are 2-combinations of the set \{a, b, c\}?

For this set, there are 6 2-permutations, but only 3 2-combinations.

Number of r-combinations

The number of r-combinations of a set with n elements is written \( \binom{n}{r} \).

\( \binom{n}{r} \) is often also written as \( \frac{n!}{r!(n-r)!} \), read "n choose r".

(\( \binom{n}{r} \)) is also called the binomial coefficient.

Theorem: \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

Proof of Theorem

What is the relationship between \( P(n, r) \) and \( \binom{n}{r} \)?

Let’s decompose \( P(n, r) \) using product rule:

First choose \( r \) elements

Then, order these \( r \) elements

How many ways to choose \( r \) elements from \( n \)?

How many ways to order \( r \) elements?

Thus, \( P(n, r) = \binom{n}{r} \times r! \)

Therefore,

\( \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)! \cdot r!} \)

Examples

We need to create a team with 5 members out of 10 candidates. How many different teams are possible?

When creating a team, we don’t care about order in which players were picked. Thus, we want \( \binom{n}{r} \), not \( P(n, r) \).

How many hands of 5 cards can be dealt from a standard deck of 52 cards?

Another Example

There are 9 faculty members in a math department, and 11 in CS department.

If we must select 3 math and 4 CS faculty for a committee, how many ways are there to form this committee?

A Corollary

Corollary: \( \binom{n}{r} = \binom{n}{n-r} \)
More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?

One More Example

- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?

Binomial Coefficients

- Recall: \( C(n, r) \) is also denoted as \( \binom{n}{r} \) and is called the binomial coefficient.
- Binomial is polynomial with two terms, e.g., \((a + b), (a + b)^2\)
- \( \binom{n}{r} \) called binomial coefficient b/c it occurs as coefficients in the expansion of \((a + b)^n\)

The Binomial Theorem

- Let \( x, y \) be variables and \( n \) a non-negative integer. Then,

\[
(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j
\]

- **Proof:** Each term in the expansion if of the form \( c \cdot x^{n-j} y^j \)
- To get such a term, must choose \( n - j \) of the \( x \)'s and \( j \) \( y \)'s
- Once you pick \( n - j \) \( x \)'s, \# of \( y \)'s determined (and vice versa)
- Since there is \( \binom{n}{n-j} = \binom{n}{j} \) ways to pick \( x^{n-j} \),
  coefficient \( c \) is \( \binom{n}{j} \)

Example

- What is the expansion of \((x + y)^4\)?

  \[
  (x + y)^4 = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}
  \]

- Therefore,

\[
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]
Another Example

- What is the coefficient of \(x^{12}y^{13}\) in the expansion of \((2x - 3y)^{25}\)?

Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.
- Corollary: \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\)

Another Corollary

- Corollary: \(\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\)

One More Corollary

- Corollary: \(\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n\)

Pascal’s Triangle

- Pascal arranged binomial coefficients as a triangle
- \(n\)’th row consists of \(\binom{n}{k}\) for \(k = 0, 1, \ldots n\)

Proof of Pascal’s Identity

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

- This identity is known as Pascal’s identity
- Proof:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!}
\]

Multiply first fraction by \(\frac{n-k+1}{n-k+1}\) and second by \(\frac{n-k+1}{n-k+1}\):

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{k 
cdot n! + (n-k+1)!(n-k+1)!}{(n-k+1)!(n-k)!}
\]
Proof of Pascal’s Identity, cont.

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!((n-k+1))!}
\]

- Factor the numerator:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{(n+1)!}{k!((n-k+1))!} = \binom{n+1}{k}
\]

- But this is exactly \(\binom{n+1}{k}\)

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Interesting Facts about Pascal’s Triangle

- What is the sum of numbers in \(n\)'th row in Pascal’s triangle (starting at \(n = 0\))?

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

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Some Fun Facts about Pascal’s Triangle, cont.

- Pascal’s triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
  - Which of the theorems we proved today explains this fact?

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Permutations with Repetitions

- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement
  - But sometimes makes sense to use an object multiple times
  - Example: How many strings of length 4 can be formed using letters in English alphabet?
    - Since string can contain same letter multiple times, we want to allow repetition!
  - A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once

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Number of Permutations with Repetition

- How many strings of length \(k\) can be formed using the 26 lower-case letters in the English alphabet?

  - Decompose using product rule:
    - How many ways to choose first letter?
    - How many ways to choose second letter?
    - ... 
    - How many possible strings?

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General Formula for Permutations with Repetition

- \(P^n(n, r)\) denotes number of \(r\)-permutations with repetition from set with \(n\) elements

- Theorem: \(P^n(n, r) = n^r\)

- Proof: \(n\) ways to select first element, \(n\) ways to select second, ... , \(n\) ways to select \(r\)'th element

  - By product rule, \(n^r\) \(r\)-permutations with repetition are possible
Examples

- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different four-digit numbers can be formed from the digits 1, 2, 3, 4?
- How many different 3-digit numbers can be formed from 1, 2, 3, 4, 5?