Review

- What is $P(n, r)$?
- What is $C(n, r)$?
- What is the inclusion-exclusion principle?

More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?
- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?

One More Example

An Example

Binomial Coefficients

- Recall: $C(n, r)$ is also denoted as $\binom{n}{r}$ and is called the binomial coefficient
- Binomial is polynomial with two terms, e.g., $(a + b), (a + b)^2$
- $(\binom{n}{r})$ called binomial coefficient b/c it occurs as coefficients in the expansion of $(a + b)^n$
The Binomial Theorem

- Let $x, y$ be variables and $n$ a non-negative integer. Then,

\[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\]

- What is the expansion of $(x + y)^4$?

Another Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.

- Corollary: \[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]

Pascal's Triangle

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<tr>
<td>$\binom{n}{0}$</td>
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<td>$\binom{n}{3}$</td>
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<td>6</td>
<td>4</td>
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Proof of Pascal’s Identity

\[\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}\]

- This identity is known as Pascal’s identity

- Proof:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}
\]

Multiply first fraction by $\frac{k}{k}$ and second by $\frac{n-k+1}{n-k+1}$:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{k!(n-k+1)!}
\]
Proof of Pascal's Identity, cont.
\[
\binom{n}{k - 1} + \binom{n}{k} = \frac{k \cdot n! + (n - k + 1)n!}{(k!)(n - k + 1)!}
\]
- Factor the numerator:
\[
\binom{n}{k - 1} + \binom{n}{k} = \frac{(n + 1)n!}{k!(n - k + 1)!} = \frac{(n + 1)!}{k!(n + 1)(n - k + 1)!}
\]
- But this is exactly \( \binom{n + 1}{k} \)

Interesting Facts about Pascal’s Triangle
- What is the sum of numbers in \( n \)’th row in Pascal’s triangle (starting at \( n = 0 \))?
- Observe: This is exactly the corollary we proved earlier!
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Some Fun Facts about Pascal’s Triangle, cont.
- Pascal’s triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
  - Why is this the case?

Permutations with Repetition
- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement
- But sometimes makes sense to use an object multiple times
- Example: How many strings of length 4 can be formed using letters in English alphabet?
  - How many different 3-digit numbers can be formed from 1, 2, 3, 4, 5?

General Formula for Permutations with Repetition
- \( P^*(n, r) \) denotes number of \( r \)-permutations with repetition from set with \( n \) elements
- What is \( P^*(n, r) \)?
- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different 3-digit numbers can be formed from 1, 2, 3, 4, 5?
Example

- An ice cream dessert consists of three scoops of ice cream.
- Each scoop can be one of the flavors: chocolate, vanilla, mint, lemon, raspberry.
- In how many different ways can you pick your dessert?
- Example of combination with repetition: “In how many ways can we pick 3 objects from 5 kinds of objects?”
- Caveat: Despite looking deceptively simple, quite difficult to figure this out (at least for me...)

Example, cont.

To solve problem, imagine we have ice cream in boxes.
- We start with leftmost box, and proceed towards right.
- At every box, you can take 0-3 scoops, and then move to next.
- Denote taking a scoop by ◦ and moving to next box by →

Example, cont.

Let’s look at some selections and their representation:

- 3 scoops of chocolate: ◦ ◦ ◦ → → → →
- 1 vanilla, 1 raspberry, 1 lemon: → ◦ → → ◦ → ◦
- 2 mint, 1 raspberry: → → ◦ ◦ → ◦ →

Invariant: r circles and n − 1 arrows (here, r = 3, n = 5)
- Our question is equivalent to: “In how many ways can we arrange r circles and n − 1 arrows?”

Result

- We’ll denote the number of ways to choose r objects from n kinds of objects \( C^r(n, r) \):
  \[
  C^r(n, r) = \binom{n + r - 1}{r}
  \]
- Example: In how many ways can we choose 3 scoops of ice cream from 5 different flavors?
  - Here, \( r = 3 \) and \( n = 5 \). Thus:
    \[
    \binom{7}{3} = \frac{7!}{3! \cdot 4!} = 35
    \]

Example 1

- Suppose there is a bowl containing apples, oranges, and pears.
  - There is at least four of each type of fruit in the bowl.
- How many ways to select four pieces of fruit from this bowl?

Example 2

- Consider a cash box containing $1 bills, $2 bills, $5 bills, $10 bills, $20 bills, $50 bills, and $100 bills.
  - There is at least five of each type of bill in the box.
- How many ways are there to select 5 bills from this cash box?
Example 3

- Assuming \(x_1, x_2, x_3\) are non-negative integers, how many solutions does \(x_1 + x_2 + x_3 = 11\) have?

Example 4

- Suppose \(x_1, x_2, x_3\) are integers s.t. \(x_1 \geq 1, x_2 \geq 2, x_3 \geq 3\).

- Then, how many solutions does \(x_1 + x_2 + x_3 = 11\) have?

Summary of Different Permutations and Combinations

<table>
<thead>
<tr>
<th>Order matters?</th>
<th>Question: How many ways to pick (r) objects from ...</th>
<th>(n) objects</th>
<th>(n) types of objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Permutation</td>
<td>(P(n, r) = \frac{n!}{(n-r)!})</td>
<td>Permutation w/ repetition (P^*(n, r) = n^r)</td>
</tr>
<tr>
<td>No</td>
<td>Combination</td>
<td>(C(n, r) = \frac{n!}{r!(n-r)!})</td>
<td>Combination w/ repetition (C^*(n, r) = \frac{(n+r-1)!}{r!(n-1)!})</td>
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Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters in the word **ALL**?

- This is not given by \(3!\) because some of the letters in this word are the same.

- Different strings: ALL, LAL, LLA \(\Rightarrow 3\) possibilities, not \(6\) because relative ordering of L’s doesn’t matter.

- In general, how can we compute the number of permutations of \(n\) objects where some of them are indistinguishable?

Proof

- Let’s decompose using product rule:
  - First, place all \(n_1\) objects of type 1
  - Then all \(n_2\) objects of type 2 etc.

- How many ways to place \(n_1\) indistinguishable objects in \(n\) slots?
Proof, cont.

▶ Now, how many ways to place \( n_2 \) objects of type 2?

▶ Continuing this way and using product rule, number of permutations is:

\[
\binom{n}{n_1} \cdot \binom{n - n_1}{n_2} \cdot \binom{n - n_1 - n_2}{n_3} \cdots \binom{n - \sum_{i=1}^{k-1} n_i}{n_k}
\]

▶ Let’s expand this definition:

\[
\frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdots \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k! \cdot (n-n_1-n_2-\cdots-n_k)!}
\]

▶ This simplifies to:

\[
\frac{n!}{n_1! n_2! \cdots n_k!}
\]

▶ Another way to see this: Compute total \( \# \) of permutations \( n! \) and then divide by \( \# \) of relative orderings between objects of type 1 \( \frac{n_1!}{n_1! n_2! \cdots n_k!} \) etc.

Example 1

▶ How many different strings can be made by ordering the letters of the word SUCCESS?

Example 2

▶ There are 3 identical red balls, 5 identical blue balls, and 2 identical green balls.

▶ In how many different ways can these balls be arranged if the first ball must be blue?

Distributing Objects into Boxes

▶ Many counting problems can be thought of as distributing objects into boxes

▶ In some cases both objects and boxes are distinguishable

▶ In some cases, boxes are distinguishable, but objects are indistinguishable

Example: Distinguishable Objects in Distinguishable Boxes

▶ How many ways are there to distribute 5 cards to each of 4 players from a deck of 52 cards?
Example, cont.

Indistinguishable Objects Into Distinguishable Boxes

- How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

Distributing Objects into Boxes Summary

- Distinguishable objects into distinguishable boxes:
  - Involves permutations

Another Example

- How many ways to distribute six distinguishable objects to five distinguishable boxes?

Example 2

- How many ways to distribute six indistinguishable objects to five distinguishable boxes?

Example 3

- How many ways to assign 15 distinguishable objects into 5 distinguishable boxes so boxes contain 1, 2, 3, 4, 5 objects respectively?