## Summary of Different Permutations and Combinations

<table>
<thead>
<tr>
<th>Order matters?</th>
<th>Question: How many ways to pick ( r ) objects from ( n ) objects with ( n ) types of objects?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Permutation ( P(n, r) = \frac{n!}{(n-r)!} )</td>
</tr>
<tr>
<td></td>
<td>Permutation w/ repetition ( P^*(n, r) = n^r )</td>
</tr>
<tr>
<td>No</td>
<td>Combination ( C(n, r) = \frac{n!}{r!(n-r)!} )</td>
</tr>
<tr>
<td></td>
<td>Combination w/ repetition ( C^*(n, r) = \frac{(n+r-1)!}{r!(n-1)!} )</td>
</tr>
</tbody>
</table>

## Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters in the word \( \text{ALL} \)?
- This is not given by \( 3! \) because some of the letters in this word are the same.
- Different strings: \( \text{ALL}, \text{LAL}, \text{LLA} \Rightarrow 3 \) possibilities, not \( 6 \) because relative ordering of L’s doesn’t matter.
- In general, how can we compute the number of permutations of \( n \) objects where some of them are indistinguishable?

## Proof

- Let’s decompose using product rule:
  - First, place all \( n_1 \) objects of type 1
  - Then all \( n_2 \) objects of type 2 etc.
  - How many ways to place \( n_1 \) indistinguishable objects in \( n \) slots?

## Permutations with Indistinguishable Objects, cont.

- Consider \( n \) objects such that:
  - \( n_1 \) of them indistinguishable of type 1
  - \( n_2 \) of them indistinguishable of type 2
  - \( \ldots \)
  - \( n_k \) of them indistinguishable of type \( k \)
- The number of permutations in this case is given by:
  \[
  \frac{n!}{n_1!n_2!\ldots n_k!}
  \]

## Proof, cont.

- Now, how many ways to place \( n_2 \) objects of type 2?
  - Continuing this way and using product rule, number of permutations is:
  \[
  \left( \frac{n}{n_1} \right) \left( \frac{n-n_1}{n_2} \right) \left( \frac{n-n_1-n_2}{n_3} \right) \cdots \left( \frac{n-\sum_{i=1}^{k-1} n_i}{n_k} \right)
  \]
Proof, cont.

\[
\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \ldots \cdot \binom{n-k-1}{n_k}
\]

Let's expand this definition:

\[
\frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdot \ldots \cdot \frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k! \cdot (n-n_1-n_2-\ldots-n_k)!}
\]

This simplifies to:

\[
\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\]

Another way to see this: Compute total # of permutations \(n!\) and then divide by \# of relative orderings between objects of type 1 \((n_1)\), \# of relative orderings of objects of type 2 \((n_2)\) etc.

Example 1

How many different strings can be made by ordering the letters of the word SUCCESS?

Example 2

There are 3 identical red balls, 5 identical blue balls, and 2 identical green balls.

In how many different ways can these balls be arranged if the first ball must be blue?

Distributing Objects into Boxes

Many counting problems can be thought of as distributing objects into boxes.

In some cases both objects and boxes are distinguishable.

In some cases, boxes are distinguishable, but objects are indistinguishable.

Example: Distinguishable Objects in Distinguishable Boxes

How many ways are there to distribute 5 cards to each of 4 players from a deck of 52 cards?
Indistinguishable Objects Into Distinguishable Boxes

- How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

Distributing Objects into Boxes Summary

- Distinguishable objects into distinguishable boxes:
  - Involves permutations
- Indistinguishable objects into distinguishable boxes:
  - Involves combination

Another Example

- How many ways to distribute six distinguishable objects to five distinguishable boxes?

Example 2

- How many ways to distribute six indistinguishable objects to five distinguishable boxes?

Example 3

- How many ways to assign 15 distinguishable objects into 5 distinguishable boxes so boxes contain 1, 2, 3, 4, 5 objects respectively?

New Topic: Graphs

- Graph is a fundamental mathematical structure in computer science
  - Graph \( G = (V, E) \) consists of a set of vertices (nodes) \( V \) and edges \( E \) between these nodes
  - Lots of applications in many areas: web search, transportation, biological models, ...
  - Will encounter graphs and graph algorithms in many different courses
Example: Social Network as a Graph
▶ Nodes represent users (Michael, Jessica, Stuart . . .)
▶ Edges represent friendship (e.g., Michael is friends with Jessica)
▶ Edge between nodes \( u \) and \( v \) is written as \((u, v)\)
▶ e.g., (Sarah, Andrew) is an edge in this graph.

Terminology
▶ Two nodes \( u \) and \( v \) are adjacent if there exists an edge between them (e.g., nodes 1 and 3)
▶ An edge \((u, v)\) is incident with nodes \( u \) and \( v \)
▶ Degree of a vertex \( v \), written \( \deg(v) \), is the number of edges incident with it
▶ Neighborhood of a vertex is the set of vertices adjacent to it

Question
Consider a graph \( G \) with vertices \( v_1, v_2, v_3, v_4 \) and edges \((v_1, v_2), (v_2, v_3), (v_1, v_3), (v_2, v_4)\).
1. Draw this graph.
2. What is the degree of each vertex?

Simple Graphs
▶ Graph contains a loop if any node is adjacent to itself
▶ A simple graph does not contain loops and there exists at most one edge between any pair of vertices
▶ Graphs that have multiple edges connecting two vertices are called multi-graphs
▶ Most graphs we will look at are simple graphs

Question
Consider a simple graph \( G \) where two vertices \( A \) and \( B \) have the same neighborhood.
Which of the following statements must be true about \( G \)?
A. The degree of each vertex must be even.
B. Both \( A \) and \( B \) have a degree of 0.
C. There cannot be an edge between \( A \) and \( B \).

Handshaking Theorem
Let \( G = (V, E) \) be a graph with \( m \) edges. Then:
\[
\sum_{v \in V} \deg(v) = 2m
\]
▶ Intuition: Each edge contributes two to the sum of the degrees
▶ Proof: By induction on the number of edges.
▶ Base case:
▶ Induction:
Applications of Handshaking Theorem

- Is it possible to construct a graph with 5 vertices where each vertex has degree 3?
- Prove that every graph has an even number of vertices of odd degree.
- If \( n \) people go to a party and everyone shakes everyone else’s hand, how many handshakes occur?

Directed Graphs

- All graphs we considered so far are undirected
- In undirected graphs, edge \((u, v)\) same as \((v, u)\)
- A directed edge (arc) is an ordered pair \((u, v)\) (i.e., \((u, v)\) not same as \((v, u)\))
- A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

- The in-degree of a vertex \( v \), written \( \deg^- (v) \), is the number of edges going into \( v \)
- \( \deg^- (a) = \)
- The out-degree of a vertex \( v \), written \( \deg^+ (v) \), is the number of edges leaving \( v \)
- \( \deg^+ (a) = \)

Handshaking Theorem for Directed Graphs

Let \( G = (V, E) \) be a directed graph. Then:

\[
\sum_{v \in V} \deg^- (v) = \sum_{v \in V} \deg^+ (v) = |E|
\]

- \( \sum_{v \in V} \deg^- (v) = \)
- \( \sum_{v \in V} \deg^+ (v) = \)

Subgraphs

- A graph \( G = (V, E) \) is a subgraph of another graph \( G' = (V', E') \) if \( V \subseteq V' \) and \( E \subseteq E' \)
- Example:

  - Graph \( G \) is a proper subgraph of \( G' \) if \( G \neq G' \).

Question

Consider a graph \( G \) with vertices \( \{v_1, v_2, v_3, v_4\} \) and edges \( (v_1, v_2), (v_1, v_3), (v_2, v_3) \).

Which of the following are subgraphs of \( G \)?

1. Graph \( G_1 \) with vertex \( v_1 \) and edge \( (v_1, v_2) \)
2. Graph \( G_2 \) with vertices \( \{v_1, v_3\} \) and edge \( (v_2, v_3) \)
3. Graph \( G_3 \) with vertices \( \{v_1, v_3\} \) and no edges
4. Graph \( G_4 \) with vertices \( \{v_1, v_2\} \) and edge \( (v_1, v_2) \)
Induced Subgraph

- Consider a graph $G = (V, E)$ and a set of vertices $V'$ such that $V' \subseteq V$.
- Graph $G'$ is the induced subgraph of $G$ with respect to $V'$ if:
  1. $G'$ contains exactly those vertices in $V'$.
  2. For all $u, v \in V'$, edge $(u, v) \in G'$ iff $(u, v) \in G$.

Subgraph induced by vertices $\{C, D\}$:

Complete Graphs

- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.

Examples Bipartite and Non-Bi-partite Graphs

- Is this graph bipartite?

- What about this graph?

Graph Coloring

- A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is $k$-colorable if it is possible to color it using $k$ colors.
  - e.g., graph on left is 3-colorable
  - Is it also 2-colorable?
- The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Question

Consider a graph $G$ with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)$.

Which of the following are valid colorings for $G$?

1. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}$
2. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}, v_4 = \text{red}$
3. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{red}, v_4 = \text{blue}$
Examples

What are the chromatic numbers for these graphs?

A -- B
C -- D

A -- B
C -- D

A -- B
C -- D

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