Announcements and Review

- Homework 6 due on Thursday
- Review: permutations, combinations
- What is binomial theorem?

Pascal’s Triangle

Pascal arranged binomial coefficients as a triangle

$n$'th row consists of $\binom{n}{k}$ for $k = 0, 1, \ldots n$

Proof of Pascal’s Identity

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

- This identity is known as Pascal’s identity

Proof:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!}
\]

Multiply first fraction by $\frac{k}{k}$ and second by $\frac{n-k+1}{n-k+1}$:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!} = \frac{n+1}{(n+1)!}
\]

But this is exactly $\binom{n+1}{k}$

Interesting Facts about Pascal’s Triangle

- What is the sum of numbers in $n$'th row in Pascal’s triangle (starting at $n = 0$)?
- Observe: This is exactly the corollary we proved earlier!

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
Some Fun Facts about Pascal’s Triangle, cont.

- Pascal’s triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
  - Which of the theorems we proved explains this fact?

Permutations with Repetitions

- Permutations allowed each object to be used only once in the arrangement, but sometimes makes sense to use an object multiple times
- Example: How many strings of length 4 can be formed using letters in English alphabet?
- A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once

General Formula for Permutations with Repetition

- \( P^*(n, r) \) denotes number of \( r \)-permutations with repetition from set with \( n \) elements
- What is \( P^*(n, r) =? \)
- Example: How many different 3-digit numbers can be formed from 1, 2, 3, 4, 5?

Combinations with Repetition

- Combinations help us to answer the question “In how many ways can we choose \( r \) objects from \( n \) objects?”
- Now, consider the slightly different question: “In how many ways can we choose \( r \) objects from \( n \) kinds of objects?”
- These questions are quite different:
  - For first question, once we pick one of the \( n \) objects, we cannot pick the same object again
  - For second question, once we pick one of the \( n \) kinds of objects, we can pick the same type of object again!
- Combination with repetition allows answering the latter type of question!

Example

- An ice cream dessert consists of three scoops of ice cream
- Each scoop can be one of the flavors: chocolate, vanilla, mint, lemon, raspberry
- In how many different ways can you pick your dessert?
- Example of combination with repetition: “In how many ways can we pick 3 objects from 5 kinds of objects?”
- Caveat: Despite looking deceptively simple, quite difficult to figure this out (at least for me...)

Example, cont.

- To solve problem, imagine we have ice cream in boxes.
- We start with leftmost box, and proceed towards right.
- At every box, you can take 0-3 scoops, and then move to next.
- Denote taking a scoop by ◦ and moving to next box by →
Let’s look at some selections and their representation:

- 3 scoops of chocolate: ◦◦◦→→→→
- 1 vanilla, 1 raspberry, 1 lemon: → ◦ o → o o
- 2 mint, 1 raspberry: →→ o o o

Invariant: \( r \) circles and \( n - 1 \) arrows (here, \( r = 3, n = 5 \))

Our question is equivalent to: "In how many ways can we arrange \( r \) circles and \( n - 1 \) arrows?"

We’ll denote the number of ways to choose \( r \) objects from \( n \) kinds of objects \( C^*(n, r) \):

\[
C^*(n, r) = \binom{n + r - 1}{r}
\]

Example: In how many ways can we choose 3 scoops of ice cream from 5 different flavors?

Here, \( r = 3 \) and \( n = 5 \). Thus:

\[
\binom{7}{3} = \frac{7!}{3! \cdot 4!} = 35
\]

Example 1

Suppose there is a bowl containing apples, oranges, and pears
- There is at least four of each type of fruit in the bowl
- How many ways to select four pieces of fruit from this bowl?

Example 2

Consider a cash box containing $1 bills, $2 bills, $5 bills, $10 bills, $20 bills, $50 bills, and $100 bills
- There is at least five of each type of bill in the box
- How many ways are there to select 5 bills from this cash box?

Example 3

Assuming \( x_1, x_2, x_3 \) are non-negative integers, how many solutions does \( x_1 + x_2 + x_3 = 11 \) have?

Example 4

Suppose \( x_1, x_2, x_3 \) are integers s.t. \( x_1 \geq 1, x_2 \geq 2, x_3 \geq 3 \).
- Then, how many solutions does \( x_1 + x_2 + x_3 = 11 \) have?
**Summary of Different Permutations and Combinations**

<table>
<thead>
<tr>
<th>Order matters?</th>
<th>Question</th>
<th>How many ways to pick r objects from n objects</th>
<th>n types of objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Permutation</td>
<td>$P(n, r) = \frac{n!}{(n-r)!}$</td>
<td>Permutation w/ repetition $P^*(n, r) = n^r$</td>
</tr>
<tr>
<td>No</td>
<td>Combination</td>
<td>$C(n, r) = \frac{n!}{r!(n-r)!}$</td>
<td>Combination w/ repetition $C^*(n, r) = \frac{(n+r-1)!}{n!(r-1)!}$</td>
</tr>
</tbody>
</table>

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**Permutations with Indistinguishable Objects**

- How many different strings can be made by reordering the letters in the word **ALL**?
- This is not given by $3!$ because some of the letters in this word are the same.
- Different strings: **ALL**, **LAL**, **LLA** ⇒ 3 possibilities, not 6 because relative ordering of L’s doesn’t matter.
- In general, how can we compute the number of permutations of $n$ objects where some of them are **indistinguishable**?

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**Proof**

- Let’s decompose using product rule:
  - First, place all $n_1$ objects of type 1
  - Then all $n_2$ objects of type 2 etc.
- How many ways to place $n_1$ indistinguishable objects in $n$ slots?
- Let’s expand this definition:
  $$\sum_{k=1}^{n-1} \frac{n!}{n_1! n_2! \cdots n_k!} \cdot \frac{(n-n_1)!}{n_1!} \cdot \frac{(n-n_1-n_2)!}{n_2!} \cdots \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_{k-1}!} \cdot \frac{n!}{n_k!}$$
- This simplifies to:
  $$\frac{n!}{n_1! n_2! \cdots n_k!}$$
- Another way to see this: Compute total # of permutations ($n!$) and then divide by # of relative orderings between objects of type 1 ($n_1!$), # of relative orderings of objects of type 2 ($n_2!$) etc.
Example 1

- How many different strings can be made by ordering the letters of the word SUCCESS?

Example 2

- There are 3 identical red balls, 5 identical blue balls, and 2 identical green balls.
- In how many different ways can these balls be arranged if the first ball must be blue?

Distributing Objects into Boxes

- Many counting problems can be thought of as distributing objects into boxes
- In some cases both objects and boxes are distinguishable
- In some cases, boxes are distinguishable, but objects are indistinguishable

Example: Distinguishable Objects in Distinguishable Boxes

- How many ways are there to distribute 5 cards to each of 4 players from a deck of 52 cards?

Indistinguishable Objects Into Distinguishable Boxes

- How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?
Distributing Objects into Boxes Summary

- Distinguishable objects into distinguishable boxes:
  - Involves permutations

- Indistinguishable objects into distinguishable boxes:
  - Involves combination

Another Example

- How many ways to distribute six distinguishable objects to five distinguishable boxes?

Example 2

- How many ways to distribute six indistinguishable objects to five distinguishable boxes?

Example 3

- How many ways to assign 15 distinguishable objects into 5 distinguishable boxes so boxes contain 1, 2, 3, 4, 5 objects respectively?