A function $f$ from a set $A$ to a set $B$ assigns each element of $A$ to exactly one element of $B$.

- $A$ is called domain of $f$, and $B$ is called codomain of $f$.
- If $f$ maps element $a \in A$ to element $b \in B$, we write $f(a) = b$.
- If $f(a) = b$, $b$ is called image of $a$; $a$ is in preimage of $b$.
- Range of $f$ is the set of all images of elements in $A$. 

### Functions Examples and Non-Examples

Is this mapping a function?

Is this mapping a function?

Is this mapping a function?

Is this mapping a function?
Function Terminology Examples

- What is the range of this function?
- What is the image of \( c \)?
- What is the preimage of \( e \)?

Image of a Set

- We can extend the definition of image to a set
- Suppose \( f \) is a function from \( A \) to \( B \) and \( S \) is a subset of \( A \)
- The image of \( S \) under \( f \) includes exactly those elements of \( B \) that are images of elements of \( S \):
  \[ f(S) = \{ t \mid \exists s \in S. t = f(s) \} \]
- What is the image of \( \{ b, c \} \)?

One-to-One Functions

- A function \( f \) is called one-to-one if and only if \( f(x) = f(y) \) implies \( x = y \) for every \( x, y \) in the domain of \( f \):
  \[ \forall x, y. (f(x) = f(y) \rightarrow x = y) \]
- One-to-one functions never assign different elements in the domain to the same element in the codomain:
  \[ \forall x, y. (x \neq y \rightarrow f(x) \neq f(y)) \]
- A one-to-one function also called injection or injective function
- Is this function one-to-one?

More Injective Function Examples

- Is this function injective?
- Consider the function \( f(x) = x^2 \) from set of integers to set of integers. Is this injective?
- What about if the domain of \( f \) is the set of non-negative integers?

Proving Injectivity Example

- Consider the function \( f \) from \( \mathbb{Z} \) to \( \mathbb{Z} \) defined as:
  \[ f(x) = \begin{cases} 
  3x + 1 & \text{if } x \geq 0 \\
  -3x + 2 & \text{if } x < 0 
  \end{cases} \]
- Prove that \( f \) is injective.
- We need to show that if \( x \neq y \), then \( f(x) \neq f(y) \)
- What proof technique do we need to use?
Onto Functions

- A function $f$ from $A$ to $B$ is called onto iff for every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$:

  $$\forall y \in B. \exists x \in A. f(x) = y$$

- Note: $\exists x \in A. \phi$ is shorthand for $\exists x. (x \in A \land \phi)$, and $\forall x \in A. \phi$ is shorthand for $\forall x. (x \in A \rightarrow \phi)$

- Onto functions also called surjective functions or surjections

- For onto functions, range and codomain are the same

Is this function onto?

Examples of Onto Functions

- Consider the function $f(x) = x^2$ from the set of integers to the set of integers. Is $f$ surjective?

Bijective Functions

- Function that is both onto and one-to-one called bijection

- Bijection also called one-to-one correspondence or invertible function

Example of bijection:

Bijection Example

- The identity function $I$ on a set $A$ is the function that assigns every element of $A$ to itself, i.e., $\forall x \in A. I(x) = x$

- Prove that the identity function is a bijection.

- Need to prove $I$ is both one-to-one and onto.

- One-to-one: We need to show $\forall x, y, (x \neq y \rightarrow I(x) \neq I(y))$

- Suppose $x \neq y$.

- Since $I(x) = x$ and $I(y) = y$, and $x \neq y$, $I(x) \neq I(y)$

Bijection Example, cont.

- Now, prove $I$ is onto, i.e., for every $b$, there exists some $a$ such that $f(a) = b$

- For contradiction, suppose there is some $b$ such that $\forall a \in A. I(a) \neq b$

- Since $I(a) = a$, this means $\forall a \in A. a \neq b$

- But since $b$ is itself in $A$, this would imply $b \neq b$, yielding a contradiction.

- Since $I$ is both onto and one-to-one, it is a bijection.
Inverse Functions

- Every bijection from set \( A \) to set \( B \) also has an inverse function.
- The inverse of bijection \( f \), written \( f^{-1} \), is the function that assigns to \( b \in B \) a unique element \( a \in A \) such that \( f(a) = b \).

Observe: Inverse functions are only defined for bijections, not arbitrary functions!
This is why bijections are also called invertible functions.

Why are Inverse Functions Only Defined on Bijections?

- Suppose \( f \) is not injective, i.e., assigns distinct elements to the same element.
- Then, the inverse is not a function because it assigns the same element to distinct elements.

Hence, inverse functions only defined for bijections!

Inverse Function Examples

- Let \( f \) be the function from \( \mathbb{Z} \) to \( \mathbb{Z} \) such that \( f(x) = x^2 \). Is \( f \) invertible?
- Let \( g \) be the function from \( \mathbb{Z} \) to \( \mathbb{Z} \) such that \( g(x) = x + 1 \). Is \( g \) invertible?

Function Composition

- Let \( g \) be a function from \( A \) to \( B \), and \( f \) from \( B \) to \( C \).
- The composition of \( f \) and \( g \), written \( f \circ g \), is defined by:
  \[
  (f \circ g)(x) = f(g(x))
  \]

Composition Example

- Let \( f \) and \( g \) be function from \( \mathbb{Z} \) to \( \mathbb{Z} \) such that \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \).
- What is \( f \circ g \)?
Another Composition Example

- Prove that \( f^{-1} \circ f = I \) where \( I \) is the identity function.
  - Since \( I(x) = x \), need to show \( (f^{-1} \circ f)(x) = x \)
  - First, \( (f^{-1} \circ f)(x) = f^{-1}(f(x)) \)
  - Let \( f(x) \) be \( y \)
  - Then, \( f^{-1}(f(x)) = f^{-1}(y) \)
  - By definition of inverse, \( f^{-1}(y) = x \) iff \( f(x) = y \)
  - Thus, \( f^{-1}(f(x)) = f^{-1}(y) = x \)

Example

- Prove that if \( f \) and \( g \) are injective, then \( f \circ g \) is also injective.

Floor and Ceiling Functions

- Two important functions in discrete math are floor and ceiling functions, both from \( \mathbb{R} \) to \( \mathbb{Z} \).
  - The floor of a real number \( x \), written \( \lfloor x \rfloor \), is the largest integer less than or equal to \( x \).

Useful Properties of Floor and Ceiling Functions

1. For integer \( n \) and real number \( x \), \( \lfloor x \rfloor = n \) iff \( n \leq x < n + 1 \)
2. For integer \( n \) and real number \( x \), \( \lceil x \rceil = m \) iff \( m - 1 < x \leq m \)
3. For any real \( x \), \( x - 1 < \lfloor x \rfloor \leq \lfloor x \rfloor < x + 1 \)

Ceiling Function

- The ceiling of a real number \( x \), written \( \lceil x \rceil \), is the smallest integer greater than or equal to \( x \).

Proofs about Floor/Ceiling Functions

Prove that \( \lceil -x \rceil = -\lfloor x \rfloor \)
Another Example

Prove that \( \lfloor x + k \rfloor = \lfloor x \rfloor + k \) where \( k \) is an integer

More Examples

Prove that \( \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor \frac{x+1}{2} \rfloor \)

Revisiting Sets

- Earlier we talked about sets and cardinality of sets
- Recall: Cardinality of a set is number of elements in that set
- This definition makes sense for sets with finitely many element, but more involved for infinite sets
- Agenda: Revisit the notion of cardinality for infinite sets

Cardinality of Infinite Sets

- Sets with infinite cardinality are classified into two classes:
  1. Countably infinite sets (e.g., natural numbers)
  2. Uncountably infinite sets (e.g., real numbers)
- A set \( A \) is called countably infinite if there is a bijection between \( A \) and the set of positive integers.
- A set \( A \) is called countable if it is either finite or countably infinite
- Otherwise, the set is called uncountable or uncountably infinite

Example

Prove: The set of odd positive integers is countably infinite.

- Need to find a function \( f \) from \( \mathbb{Z}^+ \) to the set of odd positive integers, and prove that \( f \) is bijective
- Consider \( f(n) = 2n - 1 \) from \( \mathbb{Z}^+ \) to odd positive integers
- We need to show \( f \) is bijective (i.e., one-to-one and onto)
- Let’s first prove injectivity, then surjectivity
Another Way to Prove Countable-ness

- One way to show a set $A$ is countably infinite is to give bijection between $\mathbb{Z}^+$ and $A$.
- Another way is by showing members of $A$ can be written as a sequence $(a_1, a_2, a_3, \ldots)$.
- Since such a sequence is a bijective function from $\mathbb{Z}^+$ to $A$, writing $A$ as a sequence $a_1, a_2, a_3, \ldots$ establishes one-to-one correspondence.

Another Example

Prove that the set of all integers is countable.

- We can list all integers in a sequence, alternating positive and negative integers:
  
  $a_n = 0, 1, -1, 2, -2, 3, -3, \ldots$

- Observe that this sequence defines the bijective function:
  
  $f(n) = \begin{cases} 
  n/2 & \text{if } n \text{ even} \\
  -(n - 1)/2 & \text{if } n \text{ odd}
  \end{cases}$

Rational Numbers are Countable

- Not too surprising: $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite.
- More surprising: Set of rationals is also countably infinite!
- We’ll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence.
- Recall: Every positive rational number can be written as the quotient $p/q$ of two positive integers $p, q$.

Rationals in a Table

- Now imagine placing rationals in a table such that:
  1. Rationals with $p = 1$ go in first row, $p = 2$ in second row, etc.
  2. Rationals with $q = 1$ in 1st column, $q = 2$ in 2nd column, …

Enumerating the Rationals

- How to enumerate entries in this table without missing any?
- Trick: First list those with $p + q = 2$, then $p + q = 3$, …
- Traverse table diagonally from left-to-right, in the order shown by arrows.

Enumerating the Rationals, cont.

- This allows us to list all rationals in a sequence:
  
  $1 \ 2 \ 1 \ 2 \ 3 \ 4 \ 3 \ 4 \ \ldots$

- Hence, set of rationals is countable.
Uncountability of Real Numbers

- Prime example of uncountably infinite sets is real numbers
- The fact that $\mathbb{R}$ is uncountably infinite was proven by George Cantor using the famous Cantor’s diagonalization argument
- Reminiscent of Russell’s paradox

Cantor’s Diagonalization Argument

- For contradiction, assume the set of reals was countable
- Since any subset of a countable set is also countable, this would imply the set of reals between 0 and 1 is also countable
- Now, if reals between 0 and 1 are countable, we can list them in the following way:

\[
\begin{array}{cccc}
R_1 &=& 0 & a_{11} a_{12} a_{13} \ldots a_{1n} \ldots \\
R_2 &=& 0 & a_{21} a_{22} a_{23} \ldots a_{2n} \ldots \\
R_3 &=& 0 & a_{31} a_{32} a_{33} \ldots a_{3n} \ldots \\
& \vdots & \ddots & \vdots \\
R_n &=& 0 & a_{n1} a_{n2} a_{n3} \ldots a_{nn} \ldots \\
& \vdots & \ddots & \vdots \\
\end{array}
\]

- Now, we’ll create a new real number $R$ and show that it is not equal to any of the $R_i$’s in this sequence:

- Let $R = 0.a_1 a_2 a_3 \ldots$ such that:

\[
a_i = \begin{cases} 
4 & d_{ii} \neq 4 \\
5 & d_{ii} = 4 
\end{cases}
\]

- Clearly, this new number $R$ differs from each number $R_i$ in the table in at least one digit (its $i$’th digit)

Diagonalization Argument, concluded

- Since $R$ is not in the table, this is not a complete enumeration of all reals between 0 and 1
- Hence, the set of real between 0 and 1 is not countable
- Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite