CS311H: Discrete Mathematics

Introduction to Graph Theory

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Announcements

- Homework 6 due now!
- HW7 out, due next Thursday
- Midterm 2 in two weeks!

Motivation

- Graph is a fundamental mathematical structure in computer science
- Graph $G = (V, E)$ consists of a set of vertices (nodes) $V$ and edges $E$ between these nodes
- Lots of applications in many areas: web search, networking, databases, AI . . .

Example: Social Network as a Graph

- Nodes represent users (Michael, Jessica, Stuart . . .)
- Edges represent friendship (e.g., Michael is friends with Jessica)
- Edge between nodes $u$ and $v$ is written as $(u, v)$
- e.g., (Sarah, Andrew) is an edge in this graph.

Terminology

- Two nodes $u$ and $v$ are adjacent if there exists an edge between them (e.g., nodes 1 and 3)
- An edge $(u, v)$ is incident with nodes $u$ and $v$
- Degree of a vertex $v$, written $\text{deg}(v)$, is the number of edges incident with it
- Neighborhood of a vertex is the set of vertices adjacent to it

Question

Consider a graph $G$ with vertices $v_1, v_2, v_3, v_4$ and edges $(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_2, v_4)$.

1. Draw this graph.
2. What is the degree of each vertex?
Simple Graphs

- Graph contains a loop if any node is adjacent to itself
- A simple graph does not contain loops and there exists at most one edge between any pair of vertices
- Graphs that have multiple edges connecting two vertices are called multi-graphs
- Most graphs we will look at are simple graphs

Question

Consider a simple graph $G$ where two vertices $A$ and $B$ have the same neighborhood. Which of the following statements must be true about $G$?

A. The degree of each vertex must be even.
B. Both $A$ and $B$ have a degree of 0.
C. There cannot be an edge between $A$ and $B$.

Handshaking Theorem

Let $G = (V, E)$ be a graph with $m$ edges. Then:

$$\sum_{v \in V} \text{deg}(v) = 2m$$

- Intuition: Each edge contributes two to the sum of the degrees
- Proof:

Applications of Handshaking Theorem

- Is it possible to construct a graph with 5 vertices where each vertex has degree 3?
- Prove that every graph has an even number of vertices of odd degree.
- If $n$ people go to a party and everyone shakes everyone else’s hand, how many handshakes occur?

Directed Graphs

- All graphs we considered so far are undirected
- In undirected graphs, edge $(u, v)$ same as $(v, u)$
- A directed edge (arc) is an ordered pair $(u, v)$ (i.e., $(u, v)$ not same as $(v, u)$)
- A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

- The in-degree of a vertex $v$, written $\text{deg}^-(v)$, is the number of edges going into $v$
$$\text{deg}^-(a) = \quad \text{deg}^-(a) =$$
- The out-degree of a vertex $v$, written $\text{deg}^+(v)$, is the number of edges leaving $v$
$$\text{deg}^+(a) = \quad \text{deg}^+(a) =$$
Handshaking Theorem for Directed Graphs

Let $G = (V, E)$ be a directed graph. Then:

\[ \sum_{v \in V} \text{deg}^- (v) = \sum_{v \in V} \text{deg}^+ (v) = |E| \]

Subgraphs

- A graph $G = (V, E)$ is a subgraph of another graph $G' = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$.

Example:

Graph $G$ is a proper subgraph of $G'$ if $G \neq G'$.

Induced Subgraph

- Consider a graph $G = (V, E)$ and a set of vertices $V'$ such that $V' \subseteq V$.

Graph $G'$ is the induced subgraph of $G$ with respect to $V'$ if:

1. $G'$ contains exactly those vertices in $V'$.
2. For all $u, v \in V'$, edge $(u, v) \in E'$ iff $(u, v) \in G$.

Example:

Subgraph induced by vertices $\{C, D\}$:

Bipartite graphs

- A simple undirected graph $G = (V, E)$ is called bipartite if $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in $E$ connects a $V_1$ vertex to a $V_2$ vertex.

Example:

Question

Consider a graph $G$ with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_3), (v_1, v_4), (v_2, v_3)$.

Which of the following are subgraphs of $G$?

1. Graph $G_1$ with vertex $v_3$ and edge $(v_1, v_3)$
2. Graph $G_2$ with vertices $\{v_1, v_3\}$ and no edges
3. Graph $G_3$ with vertices $\{v_1, v_2\}$ and edge $(v_1, v_2)$

Complete Graphs

- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.

Example:

How many edges does a complete graph with $n$ vertices have?
Examples Bipartite and Non-Bi-partite Graphs

▶ Is this graph bipartite?

▶ What about this graph?

Questions about Bipartite Graphs

▶ Does there exist a complete graph that is also bipartite?

▶ Consider a graph \( G \) with 5 nodes and 7 edges. Can \( G \) be bipartite?

Graph Coloring

▶ A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.

▶ A graph is \( k \)-colorable if it is possible to color it using \( k \) colors.
  - e.g., graph on left is 3-colorable
  - Is it also 2-colorable?

▶ The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Questions

Consider a graph \( G \) with vertices \( \{ v_1, v_2, v_3, v_4 \} \) and edges \( (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4) \).

Which of the following are valid colorings for \( G \)?

1. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue} \)
2. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}, v_4 = \text{red} \)
3. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{red}, v_4 = \text{blue} \)

Applications of Graph Coloring

▶ Graph coloring has lots of applications, particularly in scheduling.

▶ Example: What’s the minimum number of time slots needed so that no student is enrolled in conflicting classes?
The math department has 6 committees $C_1, \ldots, C_6$ that meet once a month.

The committee members are:

- $C_1 = \{\text{Allen, Brooks, Marg}\}$
- $C_2 = \{\text{Brooks, Jones, Morton}\}$
- $C_3 = \{\text{Allen, Marg, Morton}\}$
- $C_4 = \{\text{Jones, Marg, Morton}\}$
- $C_5 = \{\text{Allen, Brooks}\}$
- $C_6 = \{\text{Brooks, Marg, Morton}\}$

How many different meeting times must be used to guarantee that no one has conflicting meetings?

Prove that a graph $G = (V, E)$ is bipartite if and only if it is 2-colorable.

Prove that any complete graph $K_n$ has chromatic number $n$.

Theorem: A simple graph $G$ is always $\max\deg(G) + 1$-colorable. Then, $G$ is $n + 1$-colorable.
A star graph $S_n$ is a graph with one vertex $u$ at the center and the only edges are from $u$ to each of $v_1, \ldots, v_{n-1}$.

- Draw $S_2, S_3, S_4, S_5$.

What is the chromatic number of $S_n$?

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Question About Star Graphs

Suppose we have two star graphs $S_k$ and $S_m$. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph $G$?

1. The chromatic number of $G$ is 3.
2. $G$ is 2-colorable.
3. $\max\deg(G) = \max(k, m)$.