Motivation

- Graph is a fundamental mathematical structure in computer science.
- Graph \( G = (V, E) \) consists of a set of vertices (nodes) \( V \) and edges \( E \) between these nodes.
- Lots of applications in many areas: web search, transportation, biological models, etc.
- Will encounter graphs and graph algorithms in many different courses.

Example: Social Network as a Graph

- Nodes represent users (Michael, Jessica, Stuart, ...).
- Edges represent friendship (e.g., Michael is friends with Jessica).
- Edge between nodes \( u \) and \( v \) is written as \( (u, v) \).
- E.g., (Sarah, Andrew) is an edge in this graph.

Terminology

- Two nodes \( u \) and \( v \) are adjacent if there exists an edge between them (e.g., nodes 1 and 3).
- An edge \( (u, v) \) is incident with nodes \( u \) and \( v \).
- Degree of a vertex \( v \), written \( \deg(v) \), is the number of edges incident with it.
- Neighborhood of a vertex is the set of vertices adjacent to it.

Simple Graphs

- Graph contains a loop if any node is adjacent to itself.
- A simple graph does not contain loops and there exists at most one edge between any pair of vertices.
- Graphs that have multiple edges connecting two vertices are called multi-graphs.
- Most graphs we will look at are simple graphs.

Question

Consider a graph \( G \) with vertices \( v_1, v_2, v_3, v_4 \) and edges \( (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_2, v_1) \).

1. Draw this graph.
2. What is the degree of each vertex?
Question

Consider a simple graph $G$ where two vertices $A$ and $B$ have the same neighborhood.

Which of the following statements must be true about $G$?

A. The degree of each vertex must be even.
B. Both $A$ and $B$ have a degree of 0.
C. There cannot be an edge between $A$ and $B$.

Handshaking Theorem

Let $G = (V, E)$ be a graph with $m$ edges. Then:

$$\sum_{v \in V} \deg(v) = 2m$$

▶ Intuition: Each edge contributes two to the sum of the degrees
▶ Proof: By induction on the number of edges.
▶ Base case:
▶ Induction:

Applications of Handshaking Theorem

▶ Is it possible to construct a graph with 5 vertices where each vertex has degree 3?
▶ Prove that every graph has an even number of vertices of odd degree.
▶ If $n$ people go to a party and everyone shakes everyone else’s hand, how many handshakes occur?

Directed Graphs

▶ All graphs we considered so far are undirected
▶ In undirected graphs, edge $(u, v)$ same as $(v, u)$
▶ But sometimes necessary to assign directions to edges (e.g., links from one webpage to another)
▶ A directed edge (arc) is an ordered pair $(u, v)$ (i.e., $(u, v)$ not same as $(v, u)$)
▶ A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

▶ The in-degree of a vertex $v$, written $\deg^-(v)$, is the number of edges going into $v$
▶ $\deg^-(a) =$
▶ The out-degree of a vertex $v$, written $\deg^+(v)$, is the number of edges leaving $v$
▶ $\deg^+(a) =$

Handshaking Theorem for Directed Graphs

Let $G = (V, E)$ be a directed graph. Then:

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

▶ $\sum_{v \in V} \deg^-(v) =$
▶ $\sum_{v \in V} \deg^+(v) =$
Subgraphs

- A graph \( G = (V, E) \) is a subgraph of another graph \( G' = (V', E') \) if \( V \subseteq V' \) and \( E \subseteq E' \).
- Example:

Example:

Graph \( G \) is a proper subgraph of \( G' \) if \( G \neq G' \).

Induced Subgraph

- Consider a graph \( G = (V, E) \) and a set of vertices \( V' \) such that \( V' \subseteq V \).
- Graph \( G' \) is the induced subgraph of \( G \) with respect to \( V' \) if:
  1. \( G' \) contains exactly those vertices in \( V' \).
  2. For all \( u, v \in V' \), edge \((u, v)\) is in \( G' \) if \((u, v)\) is in \( G \).
- Subgraph induced by vertices \{C, D\}:

Complete Graphs

- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.

Examples Bipartite and Non-Bi-partite Graphs

- A simple undirected graph \( G = (V, E) \) is called bipartite if \( V \) can be partitioned into two disjoint sets \( V_1 \) and \( V_2 \) such that every edge in \( E \) connects a \( V_1 \) vertex to a \( V_2 \) vertex.

Question

Consider a graph \( G \) with vertices \{\(v_1, v_2, v_3, v_4\}\} and edges \((v_1, v_3), (v_1, v_4), (v_2, v_3)\).

Which of the following are subgraphs of \( G \)?

1. Graph \( G_1 \) with vertex \( v_1 \) and edge \((v_1, v_3)\)
2. Graph \( G_2 \) with vertices \{\(v_1, v_3\}\} and edge \((v_1, v_3)\)
3. Graph \( G_3 \) with vertices \{\(v_1, v_3\}\} and no edges
4. Graph \( G_4 \) with vertices \{\(v_1, v_2\}\} and edge \((v_1, v_2)\)
Graph Coloring

- A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is \( k \)-colorable if it is possible to color it using \( k \) colors.
- \( \text{e.g., graph on left is 3-colorable} \)
- Is it also 2-colorable?
- The chromatic number of a graph is the least number of colors needed to color it.

What is the chromatic number of this graph?

Question

Consider a graph \( G \) with vertices \( \{v_1, v_2, v_3, v_4\} \) and edges \((v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)\).
Which of the following are valid colorings for \( G \)?

1. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue} \)
2. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}, v_4 = \text{red} \)
3. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{red}, v_4 = \text{blue} \)

Applications of Graph Coloring

- Graph coloring has lots of applications, particularly in scheduling.
- Example: What’s the minimum number of time slots needed so that no student is enrolled in conflicting classes?

A Scheduling Problem

- The math department has 6 committees \( C_1, \ldots, C_6 \) that meet once a month.
- The committee members are:
  \( C_1 = \{\text{Allen, Brooks, Marg}\} \)
  \( C_2 = \{\text{Brooks, Jones, Morton}\} \)
  \( C_3 = \{\text{Allen, Marg, Morton}\} \)
  \( C_4 = \{\text{Jones, Marg, Morton}\} \)
  \( C_5 = \{\text{Allen, Brooks}\} \)
  \( C_6 = \{\text{Brooks, Marg, Morton}\} \)
- How many different meeting times must be used to guarantee that no one has conflicting meetings?

Bipartite Graphs and Colorability

Prove that a graph \( G = (V, E) \) is bipartite if and only if it is 2-colorable.
Complete graphs and Colorability

Prove that any complete graph $K_n$ has chromatic number $n$.

Degree and Colorability

**Theorem**: Let $G$ be a simple graph such that $\max\deg(G) = n$. Then, $G$ is $n + 1$-colorable.

Degree and Colorability, cont.

Star Graphs and Colorability

- A star graph $S_n$ is a graph with one vertex $u$ at the center and the only edges are from $u$ to each of $v_1, \ldots, v_{n-1}$.
- Draw $S_2, S_3, S_4, S_5$.
- What is the chromatic number of $S_n$?

Question About Star Graphs

Suppose we have two star graphs $S_k$ and $S_m$. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph $G$?

1. The chromatic number of $G$ is 3
2. $G$ is 2-colorable.
3. $\max\deg(G) = \max(k, m)$.