CS311H: Discrete Mathematics

Introduction to Graph Theory

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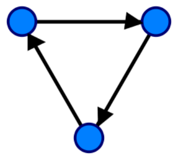
Announcements

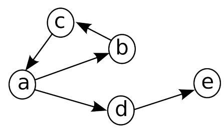
- Homeworkdue now!
- Next HW out, due next Tuesday
- Midterm 2 next Thursday!!

Directed Graphs



- All graphs we considered so far are undirected
- In undirected graphs, edge (u, v) same as (v, u)
- A directed edge (arc) is an ordered pair (u, v) (i.e., (u, v) not same as (v, u))
- A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

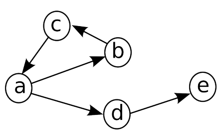


- The in-degree of a vertex v, written $\deg^{-}(v)$, is the number of edges going into v
- $\deg^{-}(a) =$
- The out-degree of a vertex v, written $\deg^{+}(v)$, is the number of edges leaving v
- $\deg^{+}(a) =$

Handshaking Theorem for Directed Graphs

Let $G = (V, E)$ be a directed graph. Then:

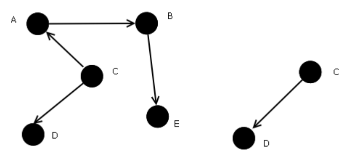
$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$



- $\sum_{v \in V} \deg^{-}(v) =$
- $\sum_{v \in V} \deg^{+}(v) =$

Subgraphs

- A graph $G = (V, E)$ is a subgraph of another graph $G' = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$
- Example:



- Graph G is a proper subgraph of G' if $G \neq G'$.

Question

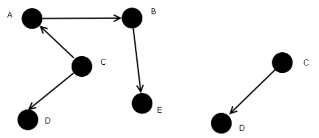
Consider a graph G with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_3), (v_1, v_4), (v_2, v_3)$.

Which of the following are subgraphs of G?

1. Graph G_1 with vertex v_1 and edge (v_1, v_3)
2. Graph G_2 with vertices $\{v_1, v_3\}$ and no edges
3. Graph G_3 with vertices $\{v_1, v_2\}$ and edge (v_1, v_2)

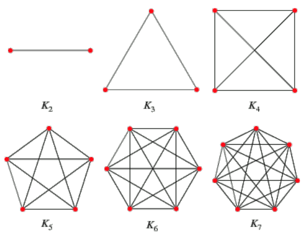
Induced Subgraph

- Consider a graph $G = (V, E)$ and a set of vertices V' such that $V' \subseteq V$
- Graph G' is the induced subgraph of G with respect to V' if:
  1. G' contains exactly those vertices in V'
  2. For all $u, v \in V'$, edge $(u, v) \in G'$ iff $(u, v) \in G$
- Subgraph induced by vertices $\{C, D\}$:



Complete Graphs

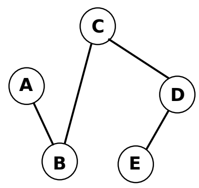
- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.



- How many edges does a complete graph with n vertices have?

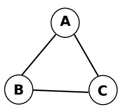
Bipartite graphs

- A simple undirected graph $G = (V, E)$ is called bipartite if V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in E connects a V_1 vertex to a V_2 vertex

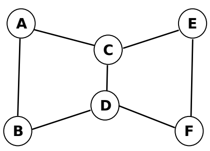


Examples Bipartite and Non-Bi-partite Graphs

- Is this graph bipartite?



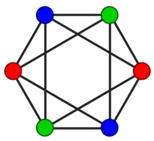
- What about this graph?



Questions about Bipartite Graphs

- Does there exist a complete graph that is also bipartite?
- Consider a graph G with 5 nodes and 7 edges. Can G be bipartite?

Graph Coloring



- A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is k-colorable if it is possible to color it using k colors.
  - e.g., graph on left is 3-colorable
  - Is it also 2-colorable?
- The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Question

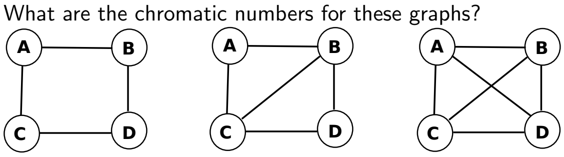
Consider a graph G with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)$.

Which of the following are valid colorings for G?

1. v_1 = red, v_2 = green, v_3 = blue
2. v_1 = red, v_2= green, v_3 = blue, v_4 = red
3. v_1 = red, v_2= green, v_3 = red, v_4 = blue

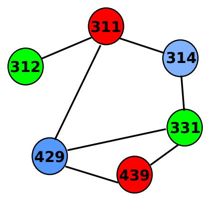
Examples

What are the chromatic numbers for these graphs?



Applications of Graph Coloring

- Graph coloring has lots of applications, particularly in scheduling.
- Example: What's the minimum number of time slots needed so that no student is enrolled in conflicting classes?



Bipartite Graphs and Colorability

Prove that a graph $G = (V, E)$ is bipartite if and only if it is 2-colorable.

Complete graphs and Colorability

Prove that any complete graph K_n has chromatic number n.

Degree and Colorability

Theorem: Every simple graph G is always $\text{max\_degree}(G) + 1$ colorable.

- Proof is by induction on the number of vertices n.
- Let $P(n)$ be the predicate "A simple graph G with n vertices is max-degree(G)-colorable"
- Base case: $n = 1$. If graph has only one node, then it cannot have any edges. Hence, it is 1-colorable.
- Induction: Consider a graph $G = (V, E)$ with $k + 1$ vertices.
- Now consider arbitrary $v \in V$ with neighbors $v_1, \ldots, v_n$

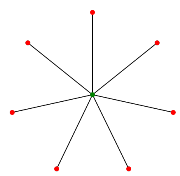
Degree and Colorability, cont.

- Remove v and all its incident edges from G; call this G'.
- By the IH, G' is $\text{max\_degree}(G') + 1$ colorable.
- Let C' be the coloring of G': Suppose C' assigns colors $c_1, \ldots, c_p$ to v's n neighbors. Clearly, $p \leq n$.
- Now, create coloring C for G:
  - $C(v') = C'(v')$ for any $v \neq v'$
  - $C(v) = c_{p+1}$

Degree and Colorability, cont.

- Two possibilities: (i) c_{p+1} was used in C', or (ii) new color
- Case 1: Then, G is $\text{max\_degree}(G') + 1$ colorable, and therefore $\text{max\_degree}(G) + 1$ colorable.
- Case 2: Coloring C uses $p + 1$ colors.
- We know $p \leq n$, where n is num neighbors
- What can we say about $\text{max\_degree}(G)$?
- Thus, $p + 1 \leq \text{max\_degree}(G) + 1$

Star Graphs and Colorability



- A star graph S_n is a graph with one vertex u at the center and the only edges are from u to each of $v_1, \ldots, v_{n-1}$.
- Draw S_4.
- What is the chromatic number of S_n?

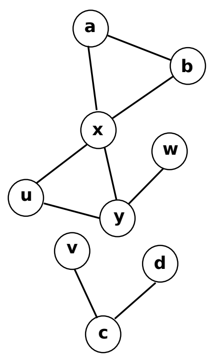
Question About Star Graphs

Suppose we have two star graphs S_k and S_m. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph G?

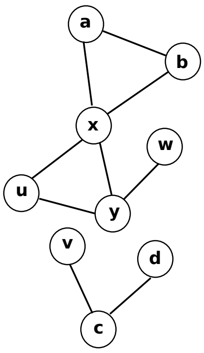
1. The chromatic number of G is 3
2. G is 2-colorable.
3. $\text{max\_degree}(G) = \max(k, m)$.

Connectivity in Graphs



- Typical question: Is it possible to get from some node u to another node v?
- Example: Train network – if there is path from u to v, possible to take train from u to v and vice versa.
- If it's possible to get from u to v, we say u and v are connected and there is a path between u and v

Paths

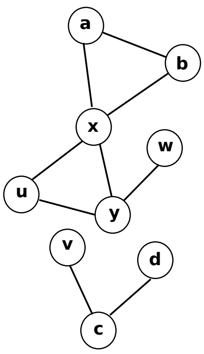


- A path between u and v is a sequence of edges that starts at vertex u, moves along adjacent edges, and ends in v.
- Example: u, x, y, w is a path, but u, y, v and u, a, x are not
- Length of a path is the number of edges traversed, e.g., length of u, x, y, w is 3
- A simple path is a path that does not repeat any edges
- u, x, y, w is a simple path but u, x, u is not

Example

- Consider a graph with vertices $\{x, y, z, w\}$ and edges $(x, y), (x, w), (x, z), (y, z)$
- What are all the simple paths from z to w?
- What are all the simple paths from x to y?
- How many paths (can be non-simple) are there from x to y?

Connectedness



- A graph is connected if there is a path between every pair of vertices in the graph
- Example: This graph not connected; e.g., no path from x to d
- A connected component of a graph G is a maximal connected subgraph of G

Example

- Prove: Suppose graph G has exactly two vertices of odd degree, say u and v. Then G contains a path from u to v.
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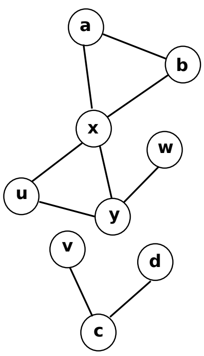
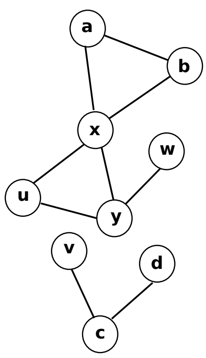
Circuits



- A circuit is a path that begins and ends in the same vertex.
- u, x, y, x, u and u, x, y, u are both circuits
- A simple circuit does not contain the same edge more than once
- u, x, y, u is a simple circuit, but u, x, y, x, u is not
- Length of a circuit is the number of edges it contains, e.g., length of u, x, y, u is 3
- In this class, we only consider circuits of length 3 or more

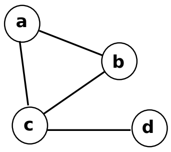
Cycles



- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, u, x, a, b, x, y, u is a circuit but not a cycle
- However, u, x, y, u is a cycle

Example

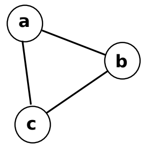
- Prove: If a graph has an odd length circuit, then it also has an odd length cycle.
- Huh? Recall that not every circuit is a a cycle.
- According to this theorem, if we can find an odd length circuit, we can also find odd length cycle.
- Example: d, c, a, b, c, d is an odd length circuit, but graph also contains odd length cycle



Proof

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Proof by strong induction on the length of the circuit.
- Base case: Length of circuit = 3.
- Only circuit of length 3 is a triangle, which is also a cycle



Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Let $P(n)$ be the predicate "If a graph has odd length circuit of length n, it also has an odd length cycle"
- Inductive step: Assume $P(3), P(5), \ldots, P(n)$ and show claim holds for $P(n+2)$
- Now, consider a circuit of length $n+2$. There are two cases:
- Case 1: Circuit is already a cycle: done!

Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.