Announcements

- Homework due now!
- Next HW out, due next Tuesday
- Midterm 2 next Thursday!!

Directed Graphs

- All graphs we considered so far are undirected
- In undirected graphs, edge \((u, v)\) same as \((v, u)\)
- A directed edge (arc) is an ordered pair \((u, v)\) (i.e., \((u, v)\) not same as \((v, u)\))
- A directed graph is a graph with directed edges

In-Degree and Out-Degree of Directed Graphs

- The in-degree of a vertex \(v\), written \(\text{deg}^- (v)\), is the number of edges going into \(v\)
- \(\text{deg}^- (a) = \)
- The out-degree of a vertex \(v\), written \(\text{deg}^+ (v)\), is the number of edges leaving \(v\)
- \(\text{deg}^+ (a) = \)

Handshaking Theorem for Directed Graphs

Let \(G = (V, E)\) be a directed graph. Then:

\[
\sum_{v \in V} \text{deg}^- (v) = \sum_{v \in V} \text{deg}^+ (v) = |E|
\]

- \(\sum_{v \in V} \text{deg}^- (v) = \)
- \(\sum_{v \in V} \text{deg}^+ (v) = \)

Subgraphs

- A graph \(G = (V, E)\) is a subgraph of another graph \(G' = (V', E')\) if \(V \subseteq V'\) and \(E \subseteq E'\)
- Example:
  - Graph \(G\) is a proper subgraph of \(G'\) if \(G \neq G'\).
Consider a graph $G$ with vertices \( \{v_1, v_2, v_3, v_4\} \) and edges \((v_1, v_3), (v_1, v_4), (v_2, v_3)\).

Which of the following are subgraphs of $G$?

1. Graph $G_1$ with vertex $v_1$ and edge $(v_1, v_3)$
2. Graph $G_2$ with vertices \( \{v_1, v_3\} \) and no edges
3. Graph $G_3$ with vertices \( \{v_1, v_2\} \) and edge $(v_1, v_2)$

### Induced Subgraph

- Consider a graph $G = (V, E)$ and a set of vertices $V'$ such that $V' \subseteq V$.
- Graph $G'$ is the induced subgraph of $G$ with respect to $V'$ if:
  1. $G'$ contains exactly those vertices in $V'$
  2. For all $u, v \in V'$, edge $(u, v) \in G'$ if $(u, v) \in G$
- Subgraph induced by vertices \( \{C, D\} \):

### Complete Graphs

- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.
- How many edges does a complete graph with $n$ vertices have?

### Bipartite graphs

- A simple undirected graph $G = (V, E)$ is called bipartite if $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in $E$ connects a $V_1$ vertex to a $V_2$ vertex.

### Examples Bipartite and Non-Bi-partite Graphs

- Is this graph bipartite?

- What about this graph?

### Questions about Bipartite Graphs

- Does there exist a complete graph that is also bipartite?

- Consider a graph $G$ with 5 nodes and 7 edges. Can $G$ be bipartite?
Graph Coloring

A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.

A graph is $k$-colorable if it is possible to color it using $k$ colors.

- e.g., graph on left is 3-colorable
- Is it also 2-colorable?
- The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Question

Consider a graph $G$ with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)$.

Which of the following are valid colorings for $G$?

1. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}$
2. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}, v_4 = \text{red}$
3. $v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{red}, v_4 = \text{blue}$

Examples

What are the chromatic numbers for these graphs?

Applications of Graph Coloring

Graph coloring has lots of applications, particularly in scheduling.

Example: What’s the minimum number of time slots needed so that no student is enrolled in conflicting classes?

Complete graphs and Colorability

Prove that any complete graph $K_n$ has chromatic number $n$. 

Bipartite Graphs and Colorability

Prove that a graph $G = (V, E)$ is bipartite if and only if it is 2-colorable.
Degree and Colorability

**Theorem:** Every simple graph \( G \) is always \( \max\text{-degree}(G) + 1 \) colorable.

- Proof is by induction on the number of vertices \( n \).
- Let \( P(n) \) be the predicate “A simple graph \( G \) with \( n \) vertices is \( \max\text{-degree}(G) \)-colorable”
- **Base case:** \( n = 1 \). If graph has only one node, then it cannot have any edges. Hence, it is 1-colorable.
- **Induction:** Consider a graph \( G = (V, E) \) with \( k + 1 \) vertices.
- Now consider arbitrary \( v \in V \) with neighbors \( v_1, \ldots, v_n \).

Degree and Colorability, cont.

- Two possibilities: (i) \( c_{p+1} \) was used in \( C' \), or (ii) new color
- **Case 1:** Then, \( G \) is \( \max\text{-degree}(G') + 1 \) colorable, and therefore \( \max\text{-degree}(G) + 1 \) colorable.
- **Case 2:** Coloring \( C \) uses \( p \) + 1 colors.
- We know \( p \leq n \), where \( n \) is num neighbors
- What can we say about \( \max\text{-degree}(G) \)?
- Thus, \( p + 1 \leq \max\text{-degree}(G) + 1 \)

Star Graphs and Colorability

- A star graph \( S_n \) is a graph with one vertex \( u \) at the center and the only edges are from \( u \) to each of \( v_1, \ldots, v_{n-1} \).
- Draw \( S_4 \).
- What is the chromatic number of \( S_4 \)?

Connectivity in Graphs

- Typical question: Is it possible to get from some node \( u \) to another node \( v \)?
- Example: Train network – if there is path from \( u \) to \( v \), possible to take train from \( u \) to \( v \) and vice versa.
- If it is possible to get from \( u \) to \( v \), we say \( u \) and \( v \) are connected and there is a path between \( u \) and \( v \).

Question About Star Graphs

Suppose we have two star graphs \( S_k \) and \( S_m \). Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph \( G \)?

1. The chromatic number of \( G \) is 3
2. \( G \) is 2-colorable.
3. \( \max\text{-degree}(G) = \max(k, m) \).
**Paths**

- A path between $u$ and $v$ is a sequence of edges that starts at vertex $u$, moves along adjacent edges, and ends in $v$.
- Example: $u, x, y, w$ is a path, but $u, a, x, y, w$ and $u, a, x$ are not.
- Length of a path is the number of edges traversed, e.g., length of $u, x, y, w$ is 3.
- A simple path is a path that does not repeat any edges.
- $u, x, y, w$ is a simple path but $u, x, u$ is not.

**Example**

- Consider a graph with vertices $\{x, y, z, w\}$ and edges $(x, y), (x, w), (x, z), (y, z)$
- What are all the simple paths from $x$ to $w$?
- What are all the simple paths from $x$ to $y$?
- How many paths (can be non-simple) are there from $x$ to $y$?

**Connectedness**

- A graph is connected if there is a path between every pair of vertices in the graph.
- Example: This graph not connected; e.g., no path from $x$ to $d$.
- A connected component of a graph $G$ is a maximal connected subgraph of $G$.

**Example**

- Prove: Suppose graph $G$ has exactly two vertices of odd degree, say $u$ and $v$. Then $G$ contains a path from $u$ to $v$.

**Circuits**

- A circuit is a path that begins and ends in the same vertex.
- $u, x, y, u$ and $u, x, y, w$ are both circuits.
- A simple circuit does not contain the same edge more than once.
- $u, x, y, u$ is a simple circuit, but $u, x, y, x, u$ is not.
- Length of a circuit is the number of edges it contains, e.g., length of $u, x, y, u$ is 3.
- In this class, we only consider circuits of length 3 or more.

**Cycles**

- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, $u, x, a, b, x, y, u$ is a circuit but not a cycle.
- However, $u, x, y, u$ is a cycle.
Example

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
- **Huh?** Recall that not every circuit is a cycle.
- **According to this theorem,** if we can find an odd length circuit, we can also find an odd length cycle.
- **Example:** $d, c, a, b, c, d$ is an odd length circuit, but graph also contains odd length cycle.

Proof

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.

- **Proof by strong induction on the length of the circuit.**
- **Base case:** Length of circuit = 3.
- **Only circuit of length 3 is a triangle, which is also a cycle.

Proof, cont.

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
  - Let $P(n)$ be the predicate “If a graph has odd length circuit of length $n$, it also has an odd length cycle.”
  - **Inductive step:** Assume $P(3), P(5), \ldots, P(n)$ and show claim holds for $P(n + 2)$.
  - **Now, consider a circuit of length $n + 2$. There are two cases:**
    - **Case 1:** Circuit is already a cycle: done!