Review

- What are properties of simple graphs?
- What is the Handshaking Theorem?
- What is the induced subgraph of $G$ on vertices $V$?

Complete Graphs

- A complete graph is a simple undirected graph in which every pair of vertices is connected by one edge.
- Complete graph with $n$ vertices denoted $K_n$.
- How many edges does a complete graph with $n$ vertices have?

Bipartite graphs

- A simple undirected graph $G = (V, E)$ is called bipartite if $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in $E$ connects a $V_1$ vertex to a $V_2$ vertex.

Examples Bipartite and Non-Bi-partite Graphs

- Is this graph bipartite?
- What about this graph?

Questions about Bipartite Graphs

- Does there exist a complete graph that is also bipartite?
- Consider a graph $G$ with 5 nodes and 7 edges. Can $G$ be bipartite?
Graph Coloring

- A coloring of a graph is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is \( k \)-colorable if it is possible to color it using \( k \) colors.
  - e.g., graph on left is 3-colorable
- The chromatic number of a graph is the least number of colors needed to color it.
  - What is the chromatic number of this graph?

Question

Consider a graph \( G \) with vertices \( \{v_1, v_2, v_3, v_4\} \) and edges \( (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4) \).

Which of the following are valid colorings for \( G \)?

1. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue} \)
2. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{blue}, v_4 = \text{red} \)
3. \( v_1 = \text{red}, v_2 = \text{green}, v_3 = \text{red}, v_4 = \text{blue} \)

Applications of Graph Coloring

- Graph coloring has lots of applications, particularly in scheduling.
  - Example: What’s the minimum number of time slots needed so that no student is enrolled in conflicting classes?

A Scheduling Problem

- The math department has 6 committees \( C_1, \ldots, C_6 \) that meet once a month.
  - The committee members are:
    - \( C_1 = \{\text{Allen, Brooks, Marg}\} \)
    - \( C_2 = \{\text{Brooks, Jones, Morton}\} \)
    - \( C_3 = \{\text{Allen, Marg, Morton}\} \)
    - \( C_4 = \{\text{Jones, Marg, Morton}\} \)
    - \( C_5 = \{\text{Allen, Brooks}\} \)
    - \( C_6 = \{\text{Brooks, Marg, Morton}\} \)
  - How many different meeting times must be used to guarantee that no one has conflicting meetings?

Bipartite Graphs and Colorability

Prove that a graph \( G = (V, E) \) is bipartite if and only if it is 2-colorable.
Complete graphs and Colorability

Prove that any complete graph $K_n$ has chromatic number $n$.

Degree and Colorability

Theorem: Every simple graph $G$ is always $\max\deg(G) + 1$ colorable.

- Proof is by induction on the number of vertices $n$.
- Let $P(n)$ be the predicate ‘A simple graph $G$ with $n$ vertices is $\max\deg(G)$-colorable’
- Base case: $n = 1$. If graph has only one node, then it cannot have any edges. Hence, it is $1$-colorable.
- Induction: Consider a graph $G = (V, E)$ with $k + 1$ vertices
  - Consider arbitrary $v \in V$ with neighbors $v_1, \ldots, v_n$

Degree and Colorability, cont.

- Remove $v$ and all its incident edges from $G$; call this $G'$
- By the IH, $G'$ is $\max\deg(G') + 1$ colorable
- Let $C'$ be the coloring of $G'$: Suppose $C'$ assigns colors $c_1, \ldots, c_p$ to $v$’s neighbors. Clearly, $p \leq n$.
- Now, create coloring $C$ for $G$:
  - $C(v') = C'(v')$ for any $v \neq v'$
  - $C(v) = c_p + 1$

Degree and Colorability, cont.

- Either $c_p + 1$ is (i) new or (ii) already used by $C$
- Case 1: If already used, $G$ is $\max\deg(G') + 1$-colorable, therefore also $\max\deg(G) + 1$-colorable
- Case 2: Coloring $C$ uses $p + 1$ colors
  - We know $p \leq n$ where $n$ is the number of $v$’s neighbors
  - What can we say about $\max\deg(G)$?
  - Thus, $G$ is $\max\deg(G) + 1$-colorable

Star Graphs and Colorability

- A star graph $S_k$ is a graph with one vertex $u$ at the center and the only edges are from $u$ to each of $v_1, \ldots, v_{k-1}$.
- Draw $S_2$, $S_3$, $S_4$, $S_5$.
- What is the chromatic number of $S_k$?

Question About Star Graphs

Suppose we have two star graphs $S_k$ and $S_m$. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph $G$?

1. The chromatic number of $G$ is 3
2. $G$ is 2-colorable.
3. $\max\deg(G) = \max(k, m)$.
Connectivity in Graphs

- Typical question: Is it possible to get from some node \( u \) to another node \( v \)?
- Example: Train network – if there is path from \( u \) to \( v \), possible to take train from \( u \) to \( v \) and vice versa.
- If it’s possible to get from \( u \) to \( v \), we say \( u \) and \( v \) are **connected** and there is a **path** between \( u \) and \( v \).

**Example**

- Consider a graph with vertices \( \{x, y, z, w\} \) and edges \( (x, y), (x, w), (x, z), (y, z) \)
- What are all the simple paths from \( z \) to \( w \)?
- What are all the simple paths from \( x \) to \( y \)?
- How many paths (can be non-simple) are there from \( x \) to \( y \)?

**Connectedness**

- A graph is **connected** if there is a path between every pair of vertices in the graph.
- Example: This graph not connected; e.g., no path from \( x \) to \( d \)
- A **connected component** of a graph \( G \) is a maximal connected subgraph of \( G \)

**Paths**

- A path between \( u \) and \( v \) is a sequence of edges that starts at vertex \( u \), moves along adjacent edges, and ends in \( v \).
- Example: \( u, x, y, w \) is a path, but \( u, y, v \) and \( u, a, x \) are not.
- Length of a path is the number of edges traversed, e.g., length of \( u, x, y, w \) is 3.
- A simple path is a path that does not repeat any edges.
- \( u, x, y, w \) is a simple path but \( u, x, u \) is not.

**Circuits**

- A circuit is a path that begins and ends in the same vertex.
- \( u, x, y, z, w \) and \( u, x, y, u \) are both circuits.
- A simple circuit does not contain the same edge more than once.
- \( u, x, y, u \) is a simple circuit, but \( u, x, y, x, u \) is not.
- Length of a circuit is the number of edges it contains, e.g., length of \( u, x, y, u \) is 3.
- In this class, we only consider circuits of length 3 or more.

Prove: Suppose graph \( G \) has exactly two vertices of odd degree, say \( u \) and \( v \). Then \( G \) contains a path from \( u \) to \( v \).
Cycles

- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, $u, x, a, b, x, y, u$ is a circuit but not a cycle.
- However, $u, x, y, u$ is a cycle.

Example

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Huh? Recall that not every circuit is a cycle.
- According to this theorem, if we can find an odd length circuit, we can also find an odd length cycle.
- Example: $d, c, a, b, c, d$ is an odd length circuit, but graph also contains odd length cycle.

Proof

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- By strong induction on the length of the circuit.
- Base case: Length of circuit $= 3$.
- Only circuit of length $3$ is a triangle, which is also a cycle.

Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Let $P(n)$ be the predicate "If a graph has an odd length circuit of length $n$, it also has an odd length cycle."
- Inductive step: Assume $P(3), P(5), \ldots, P(n)$ and show claim holds for $P(n + 2)$
- Now, consider a circuit of length $n + 2$. There are two cases:
  - Case 1: Circuit is already a cycle: done!
  - Case 2: Circuit is not a cycle, so we must have a repeated vertex in the middle:

    $C = v_0, \ldots, u_1, v_1, u_2, \ldots, v_i, u_i, \ldots, v_0$

    - We know this circuit contains two nested circuits:
      $C_1 = v_0, \ldots, u_1, v_2, u_3, \ldots, v_i, u_i, \ldots, v_0$
      $C_2 = v_1, u_2, \ldots, u_i, v_i$

    - We know that $\text{length}(C) = \text{length}(C_1) + \text{length}(C_2) = \text{odd}$
    - Means either $C_1$ or $C_2$ is odd length circuit; hence, by IH, either $C_1$ or $C_2$ contains odd length cycle.