Announcements, Review

- Homework 7 due on Thursday
- Midterm next Thursday
- Review: What is the handshaking theorem?
- What can we say about chromatic number of bipartite graph?
- What is chromatic number of $K_n$?

Degree and Colorability

**Theorem:** Every simple graph $G$ is $\max_{\text{degree}}(G) + 1$-colorable.

Star Graphs and Colorability

- A star graph $S_n$ is a graph with one vertex $u$ at the center and the only edges are from $u$ to each of $v_1, \ldots, v_{n-1}$.
- Draw $S_2, S_3, S_4, S_5$.
- What is the chromatic number of $S_n$?
Question About Star Graphs

Suppose we have two star graphs $S_k$ and $S_m$. Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph $G$?

1. The chromatic number of $G$ is 3
2. $G$ is 2-colorable.
3. $\max\deg(G) = \max(k, m)$.

Connectivity in Graphs

- Typical question: Is it possible to get from some node $u$ to another node $v$?
- Example: Train network – if there is a path from $u$ to $v$, possible to take train from $u$ to $v$ and vice versa.
- If it’s possible to get from $u$ to $v$, we say $u$ and $v$ are connected and there is a path between $u$ and $v$.

Paths

- A path between $u$ and $v$ is a sequence of edges that starts at vertex $u$, moves along adjacent edges, and ends in $v$.
- Example: $u, x, y, w$ is a path, but $u, y, v$ and $u, a, x$ are not.
- Length of a path is the number of edges traversed, e.g., length of $u, x, y, w$ is 3.
- A simple path is a path that does not repeat any edges.
- $u, x, y, w$ is a simple path but $u, x, u$ is not.
Circuits

- A circuit is a path that begins and ends in the same vertex.
- $u, x, y, z, u$ and $u, x, y, u$ are both circuits.
- A simple circuit does not contain the same edge more than once.
- $u, x, y, w$ is a simple circuit, but $u, x, y, z, u$ is not.
- Length of a circuit is the number of edges it contains, e.g., length of $u, x, y, u$ is 3.
- In this class, we only consider circuits of length 3 or more.

Proof

**Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.

- **Proof by strong induction on the length of the circuit.**

Proof, cont.

**Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.

Example

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
  - **Huh?** Recall that not every circuit is a cycle.
  - According to this theorem, if we can find an odd length circuit, we can also find odd length cycle.
  - **Example:** $d, c, a, b, c, d$ is an odd length circuit, but graph also contains odd length cycle.

Cycles

- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, $u, x, a, b, x, y, u$ is a circuit but not a cycle.
- However, $u, x, y, u$ is a cycle.

Proof, cont.

**Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
**Colorability and Cycles**

*Prove:* If a graph is 2-colorable, then all cycles are of even length.

**Example**

Is this graph 2-colorable?

**Distance Between Vertices**

- The distance between two vertices \( u \) and \( v \) is the length of the shortest path between \( u \) and \( v \).
- What is the distance between \( u \) and \( b \)?
- What is the distance between \( u \) and \( x \)?
- What is the distance between \( x \) and \( w \)?

**More Colorability and Cycles**

*Prove:* If graph has no odd length cycles, then graph is 2-colorable.

**The Algorithm**

- Pick any vertex \( v \) in the graph.
- If a vertex \( u \) has odd distance from \( v \), color it blue.
- Otherwise, color it red.

**Proof**

- We will now prove: "If the graph does not have odd length cycles, the algorithm is correct."
- Correctness of the algorithm implies graph is 2-colorable.
- Proof by contradiction.
- Suppose graph does not have odd length cycles, but the algorithm produces an invalid coloring.
- Means there exist two vertices \( x \) and \( y \) that are assigned the same color.
Proof, cont.

- **Case 1**: They are both assigned red
  - We know $n, m$ are both even
  - This means we now have an odd-length circuit involving $n, m$
  - By theorem from earlier, this implies that graph has odd length cycle, i.e., contradiction
  - Case 2 is exactly the same.

Putting It All Together

- **Theorem**: A graph is 2-colorable if and only if it does not have odd-length cycles
- **Corollary**: A graph is bipartite if and only if it does not have odd-length cycles
- **Example**: Consider a graph $G$ with vertices $a, b, c, d, e, f$
  - Is $G$ bipartite if its edges are $(a, b), (a, c), (a, d), (a, e), (a, f)$?
  - What about if its edges are $(a, f), (e, f), (e, d), (e, c), (a, c)$?