Review

- What does it mean for a graph to be bipartite?
- What is the chromatic number of a graph?
- How do you prove that the chromatic number is $n$?

Bipartite Graphs and Colorability

Prove that a graph $G = (V,E)$ is bipartite if and only if it is 2-colorable.

Complete graphs and Colorability

Prove that any complete graph $K_n$ has chromatic number $n$.

Degree and Colorability

**Theorem:** Let $G$ be a simple graph such that $\max\deg(G) = n$. Then, $G$ is $n + 1$-colorable.
Example

Consider a graph with vertices \( \{x, y, z, w\} \) and edges \((x, y), (x, w), (x, z), (y, z)\).

- What are all the simple paths from \( z \) to \( w \)?
- What are all the simple paths from \( x \) to \( y \)?
- How many paths (can be non-simple) are there from \( x \) to \( y \)?

Question About Star Graphs

Suppose we have two star graphs \( S_k \) and \( S_m \). Now, pick a random vertex from each graph and connect them with an edge.

Which of the following statements must be true about the resulting graph \( G \)?

1. The chromatic number of \( G \) is 3.
2. \( G \) is 2-colorable.
3. \( \text{max degree}(G) = \max(k, m) \).

Star Graphs and Colorability

- A star graph \( S_n \) is a graph with one vertex \( u \) at the center and the only edges are from \( u \) to each of \( v_1, \ldots, v_n \).
- Draw \( S_2, S_3, S_4, S_5 \).
- What is the chromatic number of \( S_n \)?

Connectivity in Graphs

- Typical question: Is it possible to get from some node \( u \) to another node \( v \)?
- Example: Train network – if there is path from \( u \) to \( v \), possible to take train from \( u \) to \( v \) and vice versa.
- If it’s possible to get from \( u \) to \( v \), we say \( u \) and \( v \) are connected and there is a path between \( u \) and \( v \).

Paths

- A path between \( u \) and \( v \) is a sequence of edges that starts at vertex \( u \), moves along adjacent edges, and ends in \( v \).
- Example: \( u, x, y, w \) is a path, but \( u, y, v \) and \( u, a, x \) are not.
- Length of a path is the number of edges traversed, e.g., length of \( u, x, y, w \) is 3.
- A simple path is a path that does not repeat any edges.
- \( u, x, y, w \) is a simple path but \( u, x, x \) is not.

Connectedness

- A graph is connected if there is a path between every pair of vertices in the graph.
- Example: This graph not connected; e.g., no path from \( x \) to \( d \).
- A connected component of a graph \( G \) is a maximal connected subgraph of \( G \).
Example

- **Prove:** Suppose graph $G$ has exactly two vertices of odd degree, say $u$ and $v$. Then $G$ contains a path from $u$ to $v$.

Example

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
  - **Huh?** Recall that not every circuit is a cycle.
  - **According to this theorem,** if we can find an odd length circuit, we can also find odd length cycle.
  - **Example:** $d, c, a, b, c, d$ is an odd length circuit, but graph also contains odd length cycle.

Proof

**Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.

- **Proof by strong induction on the length of the circuit.**

Proof, cont.

**Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.

- **In this class, we only consider circuits of length 3 or more.**

Cycles

- **A cycle** is a simple circuit with no repeated vertices other than the first and last ones.
  - **For instance,** $u, x, a, b, x, u$ is a circuit but not a cycle.
  - **However,** $u, x, y, u$ is a cycle.

Circuits

- **A circuit** is a path that begins and ends in the same vertex.
  - $u, x, y, x, u$ and $u, x, y, u$ are both circuits.
  - **A simple circuit** does not contain the same edge more than once.
  - $u, x, y, u$ is a simple circuit, but $u, x, y, x, u$ is not.
  - Length of a circuit is the number of edges it contains, e.g., length of $u, x, y, u$ is 3.
  - In this class, we only consider circuits of length 3 or more.
Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

Example

Is this graph 2-colorable?

More Colorability and Cycles

Prove: If graph has no odd length cycles, then graph is 2-colorable.

The Algorithm

Pick any vertex \( v \) in the graph.

If a vertex \( u \) has odd distance from \( v \), color it blue.

Otherwise, color it red.

Colorability and Cycles

Prove: If a graph is 2-colorable, then all cycles are of even length.

Distance Between Vertices

The distance between two vertices \( u \) and \( v \) is the length of the shortest path between \( u \) and \( v \).

What is the distance between \( u \) and \( b \)?

What is the distance between \( u \) and \( x \)?

What is the distance between \( x \) and \( w \)?
Proof

- We will now prove: “If the graph does not have odd length cycles, the algorithm is correct.”
- Correctness of the algorithm implies graph is 2-colorable.
- Proof by contradiction.
- Suppose graph does not have odd length cycles, but the algorithm produces an invalid coloring.
- Means there exist two vertices $x$ and $y$ that are assigned the same color.

Proof, cont.

- Case 1: They are both assigned red

- We know $n$, $m$ are both even
- This means we now have an odd-length circuit involving $n$, $m$
- By theorem from earlier, this implies that graph has odd length cycle, i.e., contradiction
- Case 2 is exactly the same.

Putting It All Together

- Theorem: A graph is 2-colorable if and only if it does not have odd-length cycles
- Corollary: A graph is bipartite if and only if it does not have odd-length cycles
- Example: Consider a graph $G$ with vertices $a, b, c, d, e, f$
  - Is $G$ partite if its edges are $(a, f), (e, f), (e, d), (e, d), (a, e)$?