Announcements, Review

- Midterm on Thursday – arrive at least 5 minutes early!
- Can bring 3 cheat sheets
- Review: What is a tree? What is a forest?
- How many edges does a tree with \( n \) nodes have?
- What is a rooted tree?

\textbf{m-ary Trees}

- A rooted tree is called an \textit{m-ary tree} if every vertex has no more than \( m \) children.
- An \textit{m-ary tree} where \( m = 2 \) is called a \textit{binary tree}.
- A \textit{full m-ary tree} is a tree where every internal node has exactly \( m \) children.
- Which are full binary trees?

\textbf{Useful Theorem}

\textbf{Theorem:} An \textit{m-ary tree} of height \( h \geq 1 \) contains at most \( m^h \) leaves.

- Proof is by induction on height \( h \).

\textbf{Questions}

- What is maximum number of leaves in binary tree of height 5?
- If binary tree has 100 leaves, what is a lower bound on its height?
- If binary tree has 2 leaves, what is an upper bound on its height?
Balanced Trees

- An m-ary tree is balanced if all leaves are at levels $h$ or $h - 1$

- "Every full tree must be balanced." – true or false?
- "Every balanced tree must be full." – true or false?

Theorem about Full and Balanced Trees

Theorem: For a full and balanced m-ary tree with height $h$ and $n$ leaves, we have $h = \lceil \log_m n \rceil$

Planar Graphs

- A graph is called planar if it can be drawn in the plane without any edges crossing (called planar representation).

- Is this graph planar?

- In this class, we will assume that every planar graph has at least 3 edges.

A Non-planar Graph

- The complete graph $K_5$ is not planar:

- Why can $K_5$ not be drawn without any edges crossing?

Regions of a Planar Graph

- The planar representation of a graph splits the plane into regions (sometimes also called faces):

- Every planar graph has an outer region, which is unbounded.

- Degree of a region $R$, written $\deg(R)$, is the number of edges bordering $R$.

Examples

- How many regions does this graph have?

- What is the degree of its outer region?

- How many regions does a graph have if it has no cycles?

- Given a planar simple graph, what is the minimum degree a region can have?

- What is the relationship between $\sum \deg(R)$ and the number of edges?
Euler’s Formula

Euler’s Formula: Let \( G = (V, E) \) be a planar connected graph with regions \( R \). Then, the following formula always holds:

\[
|R| = |E| - |V| + 2
\]

All planar representations of a graph split the plane into the same number of regions!

Proof of Euler’s Formula

▶ Proof is by induction on the number of regions \( |R| \)
▶ Base case: \( |R| = 1 \). i.e., \( G \) does not have cycles (i.e., a tree)

Proof, cont.

▶ Induction: Assume Euler’s formula holds for graph with \( k \) regions, show it for graph with \( k + 1 \) regions

An Application of Euler’s Formula

▶ Suppose a connected planar simple graph \( G \) has 6 vertices, each with degree 4.
▶ How many regions does a planar representation of \( G \) have?

A Corollary of Euler’s Formula

Theorem: Let \( G \) be a connected planar simple graph with \( v \) vertices and \( e \) edges. Then \( e \leq 3v - 6 \)

Why is this Theorem Useful?

Theorem: Let \( G \) be a connected planar simple graph with \( v \) vertices and \( e \) edges. Then \( e \leq 3v - 6 \)

▶ Can be used to show graph is not planar.
▶ Example: Prove that \( K_5 \) is not planar.
Seven Bridges of Königsberg

- Town of Königsberg in Germany divided into four parts by the Pregel river and had seven bridges
- Townspeople wondered if one can start at point A, cross all bridges exactly once, and come back to A
- Mathematician Euler heard about this puzzle and solved it

Euler Circuits and Euler Paths

- Given graph $G$, an Euler circuit is a simple circuit containing every edge of $G$.
- Euler path is a simple path containing every edge of $G$.

Examples

- Does this graph have a Euler circuit or path?

Examples

- What about this one?

Examples

- Are there some criteria that allow us to easily determine if a graph has Euler circuit or path?

Theorem about Euler Circuits

Theorem: A connected multigraph $G$ with at least two vertices contains an Euler circuit if and only if each vertex has even degree.

Proof of Sufficiency

- Now, prove the if part – much more difficult!
- By strong induction on the number of edges $e$
- Base case: $e = 2$
- Induction: Suppose claim holds for every graph with $\leq e$ edges; show it holds for graph with $e + 1$ edges
- Consider graph $G$ with $e + 1$ edges and where every vertex has even degree
- Observe: $G$ cannot be a tree – why?
Proof, cont.

Revisiting Example

- Does this graph have an Euler circuit?

An Euler circuit:

- Does this have an Euler circuit?

Necessary and Sufficient Conditions for Euler Paths

Theorem: A connected multigraph $G$ contains an Euler path iff there are exactly 0 or 2 vertices of odd degree.

- Let’s first prove necessity: Suppose $G$ has Euler path $P$ with start and end-points $u$ and $v$

Proof of Sufficiency

- Suppose $G$ has exactly 0 or 2 vertices with odd degree
  - Case 1: If no vertices with odd degree, must have Euler circuit
  - Case 2: It has exactly two vertices, say $u$, $v$, with odd degree

Example

- Does this graph have Euler path?

Graph with an Euler path: