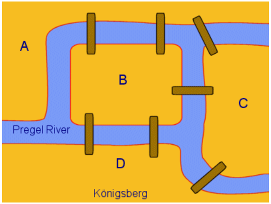
CS311H: Discrete Mathematics

More Graph Theory

Instructor: Işıl Dillig

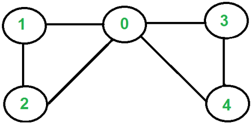
Seven Bridges of Königsberg



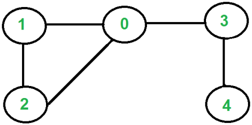
- Town of Königsberg in Germany divided into four parts by the Pregel river and had seven bridges
- Townspeople wondered if one can start at point A, cross all bridges exactly once, and come back to A
- Mathematician Euler heard about this puzzle and solved it

Euler Circuits and Euler Paths

- Given graph G, an Euler circuit is a simple circuit containing every edge of G.

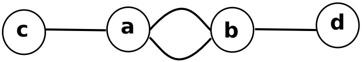


- Euler path is a simple path containing every edge of G.

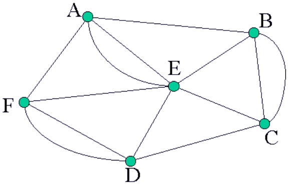


Examples

- Does this graph have a Euler circuit or path?



- What about this one?

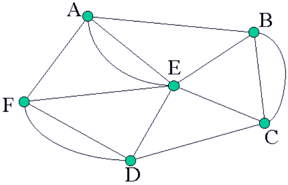


- Are there some criteria that allow us to easily determine if a graph has Euler circuit or path?

Theorem about Euler Circuits

Theorem: A connected multigraph G with at least two vertices contains an Euler circuit if and only if each vertex has even degree.

- We can immediately tell if graph has Euler circuit by looking at the degree of vertices!
- Does this graph have Euler circuit?



Proof of Necessity

- Let's first prove necessity: If G has Euler circuit, then every vertex has even degree.
- For contradiction, suppose some vertex v has odd degree
- Label the i'th edge used to visit v as e_i^v
- What can we say about the last edge e_n^v used to visit v?

Necessity proof, cont.

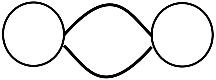
- Case 1: v is the start vertex for circuit $\Rightarrow$ We leave v for the first time using e_1^v
- Every time we leave v, we use an odd numbered edge; every time we enter, use even-numbered edge
- What does this imply about last edge e_n^v that we use to visit v?

Necessity proof, cont.

- Case 2: v is not the start vertex for circuit $\Rightarrow$ We enter v for the first time using e_1^v
- Every time we enter v, we use an odd numbered edge; every time we exit, use even-numbered edge
- What does this imply about last edge e_n^v that we use to visit v?

Proof of Sufficiency

- If every vertex has even degree, then G has Euler circuit
- By strong induction on the number of edges e
- Base case: $e = 2$



- Induction: Suppose claim holds for every graph with $\leq e$ edges; show it holds for graph with $e + 1$ edges
- Consider graph G with $e + 1$ edges and where every vertex has even degree
- Observe: G cannot be a tree – why?

Proof, cont.

- This means G must contain a cycle, say C
- Now, remove all edges in C from G to obtain graph G'
- G' may not be connected, suppose it consists of connected components $G_1, \ldots, G_n$
- Each vertex in a cycle has exactly two adjacent edges that are part of the cycle
- Hence, if all nodes in G have even degree, then nodes in each G_i must also have even degree

Proof, cont.

- Now, each G_i is connected and every vertex has even degree
- By IH, each G_i has an Euler circuit, say C_i
- We can now also build an Euler circuit for G using these C_i's
- Start at some vertex v in C, traverse along C until we reach a vertex v_i in connected component G_i
- Now, traverse C_i and come back to v_i
- Continue until we are back at v_i
- This is an Euler circuit because we've traversed every edge and haven't repeated any edges

Euler Paths

- We have a necessary and sufficient condition for Euler circuits
- What about similar conditions for Euler paths?

Necessary and Sufficient Conditions for Euler Paths

Theorem: A connected multigraph G contains an Euler path iff there are exactly 0 or 2 vertices of odd degree.

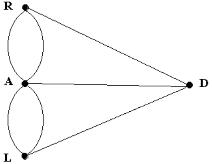
- Let's first prove necessity: Suppose G has Euler path P with start and end-points u and v
- Case 1: u, v are the same – then P is an Euler circuit, hence it must have 0 vertices of degree
- Case 2: u, v are distinct
- Except for u, v, we must enter and leave each vertex same number of times – these must have even degree
- We must leave u one more time than we enter it, and we enter v one more time than we leave it, so they have odd degree

Proof of Sufficiency

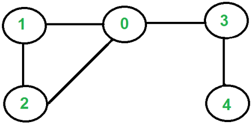
- Suppose G has exactly 0 or 2 vertices with odd degree
- Case 1: If no vertices with odd degree, must have Euler circuit
- Case 2: It has exactly two vertices, say u, v, with odd degree
- Now, add an edge between u, v to generate graph G'
- All vertices in G' have even degree – so G' has Euler circuit
- This means G has Euler path with start and end-points u, v

Example

- Does this graph have Euler path?

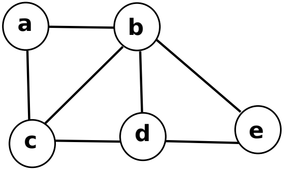


- Graph with an Euler path:



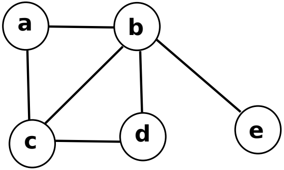
Hamilton Paths and Circuits

- A Hamilton circuit in a graph G is a simple circuit that visits every vertex in G exactly once (except the start node).



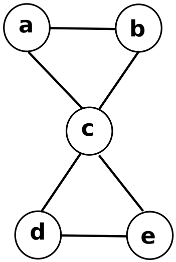
- Note that all Hamilton circuits are cycles!

- A Hamilton path in a graph G is a simple path that visits every vertex in G exactly once.



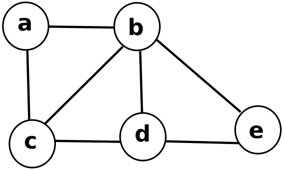
Are All Euler Circuits Also Hamilton Circuits?

- Not every Euler circuit is a Hamilton circuit:



Are All Hamilton Circuits Euler Circuits?

- Not every Hamilton circuit is an Euler circuit:



- Unlike Euler circuits, no necessary and sufficient criteria for identifying Hamilton circuits or paths