Structural Induction

Last time, we talked about recursively defined structures like sets and strings.

Structural induction is a technique that allows us to apply induction on recursive definitions even if there is no integer.

Structural induction is also no more powerful than regular induction, but can make proofs much easier.

Structural Induction Overview

- Suppose we have:
  - a recursively defined structure $S$
  - a property $P$ we'd like to prove about $S$
- Structural induction works as follows:
  1. Base case: Prove $P$ about base case in recursive definition
  2. Inductive step: Assuming $P$ holds for sub-structures used in the recursive step of the definition, show that $P$ holds for the recursively constructed structure.

Example 1

- Consider the following recursively defined set $S$:
  1. $a \in S$
  2. If $x \in S$, then $(x) \in S$
- Prove by structural induction that every element in $S$ contains an equal number of right and left parentheses.
  - Base case: $a$ has 0 left and 0 right parentheses
  - Inductive step: By the inductive hypothesis, $x$ has equal number, say $n$, of right and left parentheses.
  - Thus, $(x)$ has $n+1$ left and $n+1$ right parentheses.

Example 2

- Consider the set $S$ defined recursively as follows:
  - Base case: $3 \in S$
  - Recursive step: If $x \in S$ and $y \in S$, then $x + y \in S$
- Prove $S$ is set of all positive integers that are multiples of 3
  - Let $A$ be the set of all positive integers divisible by 3
  - We want to show that $A = S$
  - To do this, we need to prove $S \subseteq A$ and $A \subseteq S$

Proof, Part I

Consider the set $S$ defined recursively as follows: $3 \in S$ and if $x \in S$ and $y \in S$, then $x + y \in S$

- Let's first prove $S \subseteq A$, i.e., any element in $S$ is divisible by 3
  - Base case:
  - Inductive step:
Proof, Part II

- Next, need to show $S$ includes all positive multiples of 3
- Therefore, need to prove that $3n \in S$ for all $n \geq 1$
- We’ll prove this by induction on $n$:
  - Base case ($n=1$):
  - Inductive hypothesis:
  - Need to show:

Proving Correctness of Reverse

- Earlier, we defined a $\text{reverse}(w)$ function for length of strings:
  - Base case: $\text{reverse}(\epsilon) = \epsilon$
  - Recursive step: $\text{reverse}(wa) = a \cdot \text{reverse}(w)$ where $w \in \Sigma^*$ and $a \in \Sigma$
- Prove $\forall y, x \in \Sigma^*$. $\text{reverse}(xy) = \text{reverse}(y) \cdot \text{reverse}(x)$
- Let $P(y)$ be the property
  $\forall x \in \Sigma^*$: $\text{reverse}(xy) = \text{reverse}(y) \cdot \text{reverse}(x)$
- We’ll prove by structural induction that $\forall y \in \Sigma^*$. $P(y)$ holds

Proof of Correctness of Reverse, cont.

$P(y) : \forall x \in \Sigma^*. \text{reverse}(xy) = \text{reverse}(y) \cdot \text{reverse}(x)$

- Inductive step: $y = za$ where $z \in \Sigma^*$ and $a \in \Sigma$
- Want to show: $\text{reverse}(zaa) = \text{reverse}(za) \cdot \text{reverse}(x)$
- $\text{reverse}(zaa) = a \cdot \text{reverse}(za)$
- By the inductive hypothesis, $\text{reverse}(za) = \text{reverse}(z) \cdot \text{reverse}(a)$
- Thus, $a \cdot \text{reverse}(za) = a \cdot \text{reverse}(z) \cdot \text{reverse}(a)$
- By definition, $a \cdot \text{reverse}(za) = \text{reverse}(za)$
- Hence, $\text{reverse}(zaa) = \text{reverse}(za) \cdot \text{reverse}(x)$

One More Reverse Example

- Prove that $\text{reverse}(\text{reverse}(s)) = s$
- We’ll prove this by structural induction
- But need previous lemma for the proof to go through!
Properties of Length

- Prove the following property about the length function:
  \[ \forall y, x \in \Sigma^*. \; len(xy) = len(x) + len(y) \]

Generalized Induction

- Can use induction to prove properties of any well-ordered set:
  - Base case: Prove property about least element in set
  - Inductive step: To prove \( P(\epsilon) \) for all \( \epsilon' < \epsilon \)
  - Mathematical induction is just a special case of this

Generalized Induction Example

- Suppose that \( a_{m,n} \) is defined recursively for \( (m, n) \in \mathbb{N} \times \mathbb{N} \):
  \[
  \begin{align*}
  a_{0,0} &= 0 \\
  a_{m,n} &= \begin{cases} 
  a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\
  a_{m,n-1} + n & \text{if } n > 0
  \end{cases}
  \end{align*}
  \]
  - Show that \( a_{m,n} = m + n(n+1)/2 \)
  - Proof is by induction on \( (m, n) \) where \( (m, n) \in (\mathbb{N} \times \mathbb{N}, \preceq) \)
  - Base case:
    - By recursive definition, \( a_{0,0} = 0 \)
    - \( 0 + 0 \cdot 1/2 = 0 \), thus, base case holds.

Ordered Pairs of Natural Numbers

- Consider the set \( \mathbb{N} \times \mathbb{N} \), pairs of non-negative integers
- Let’s define the following order \( \preceq \) on this set:
  \[ (x_1, y_1) \preceq (x_2, y_2) \iff \begin{cases} 
  x_1 < x_2 \\
  x_1 = x_2 \land y_1 \leq y_2
  \end{cases} \]
- This is an example of lexicographic order, which is a kind of total order
- Therefore, \( (\mathbb{N} \times \mathbb{N}, \preceq) \) is a well-ordered set
- Question: What is the least element of this set?
Inductive Step

Show \( a_{m,n} = m + n(n + 1)/2 \) for:

\[
\begin{align*}
a_{0,0} &= 0 \\
a_{m,n} &= \begin{cases} 
  a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\
  a_{m,n-1} + n & \text{if } n > 0
\end{cases}
\end{align*}
\]

- **Inductive hypothesis:** For all \((0,0) \leq (i,j) < (k_1,k_2)\):
  \( a_{i,j} = i + \frac{j(j+1)}{2} \)
- **Want to show:**
  \( a_{k_1,k_2} = k_1 + \frac{k_2(k_2+1)}{2} \)

Example, cont.

Show \( a_{m,n} = m + n(n + 1)/2 \) for:

\[
\begin{align*}
a_{0,0} &= 0 \\
a_{m,n} &= \begin{cases} 
  a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\
  a_{m,n-1} + n & \text{if } n > 0
\end{cases}
\end{align*}
\]

- **Case 1:** \( k_2 = 0, k_1 > 0 \). Then, \( a_{k_1,k_2} = a_{k_1-1,k_2} + 1 \)
  - Since \((k_1-1, k_2) < (k_1, k_2)\), inductive hypothesis applies.
  - By the IH, we know:
    \( a_{k_1-1,k_2} = k_1 - 1 + \frac{k_2(k_2+1)}{2} \)
  - But then \( a_{k_1,k_2} = a_{k_1-1,k_2} + 1 = k_1 + \frac{k_2(k_2+1)}{2} \)

Another Example

- Consider the function \( \mathbb{Z} \to \mathbb{Z} \) defined recursively as follows:
  \[
  f(-1) = -1 \\
  f(n) = f(n+1) + n \quad \text{for } n < -1
  \]
- Prove that:
  \[
  f(n) = \frac{|n| \cdot (|n| + 1)}{2}
  \]