CS311H: Discrete Mathematics

Number Theory

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Review

- What does it mean for two ints a, b to be congruent mod m?
- What is the Division theorem?
- If $a|b$ and $a|c$, does it mean $b|c$?
- What is the Fundamental Theorem of Arithmetic?

Computing GCDs

- Simple algorithm to compute gcd of a, b:
  - Factorize a as $p_1^{i_1} p_2^{i_2} \ldots p_n^{i_n}$
  - Factorize b as $p_1^{j_1} p_2^{j_2} \ldots p_n^{j_n}$
  - $\gcd(a, b) = p_1^{min(i_1, j_1)} p_2^{min(i_2, j_2)} \ldots p_n^{min(i_n, j_n)}$
- But this algorithm is not good because prime factorization is computationally expensive! (not polynomial time)
- Much more efficient algorithm to compute gcd, called the Euclidian algorithm

Insight Behind Euclid's Algorithm

- Theorem: Let $a = bq + r$. Then, $\gcd(a, b) = \gcd(b, r)$
- e.g., Consider $a = 12$, $b = 8$ and $a = 12, b = 5$
- Proof: We'll show that a, b and b, r have the same common divisors – implies they have the same gcd.

⇒ Suppose d is a common divisor of a, b, i.e., $d|a$ and $d|b$

- By theorem we proved earlier, this implies $d|a - bq$
- Since $a - bq = r$, $d|r$. Hence d is common divisor of b, r.

⇐ Now, suppose $d|b$ and $d|r$. Then, $d|bq + r$

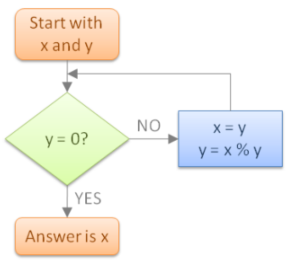
- Hence, $d|a$ and d is common divisor of a, b □

Using this Theorem

Theorem: Let $a = bq + r$. Then, $\gcd(a, b) = \gcd(b, r)$

- Suggests following recursive strategy to compute gcd(a, b):
  - Base case: If b is 0, then gcd is a
  - Recursive case: Compute gcd(b, a $\bmod$ b)
- Claim: We'll eventually hit base case – why?

Euclidian Algorithm



- Find gcd of 72 and 20
- $12 = 72\%20$
- $8 = 20\%12$
- $4 = 12\%8$
- $0 = 8\%4$
- gcd is 4!

GCD as Linear Combination

- $\gcd(a, b)$ can be expressed as a linear combination of a and b
- Theorem: If a and b are positive integers, then there exist integers s and t such that:

$$\gcd(a, b) = s \cdot a + t \cdot b$$

- Furthermore, Euclidian algorithm gives us a way to compute these integers s and t (known as extended Euclidian algorithm)

Example

- Express $\gcd(72, 20)$ as a linear combination of 72 and 20
- First apply Euclid's algorithm (write $a = bq + r$ at each step):
  1. $72 = 3 \cdot 20 + 12$
  2. $20 = 1 \cdot 12 + 8$
  3. $12 = 1 \cdot 8 + 4$
  4. $8 = 2 \cdot 4 + 0 \Rightarrow$ gcd is 4
- Now, using (3), write 4 as $12 - 1 \cdot 8$
- Using (2), write 4 as $12 - 1 \cdot (20 - 1 \cdot 12) = 2 \cdot 12 - 1 \cdot 20$
- Using (1), we have $12 = 72 - 3 \cdot 20$, thus:

$$4 = 2 \cdot (72 - 3 \cdot 20) - 1 \cdot 20 = 2 \cdot 72 + (-7) \cdot 20$$

Exercise

Use the extended Euclid algorithm to compute $\gcd(38, 16)$.

A Useful Result

- Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.
- Proof: Since a, b are relatively prime $\gcd(a, b) = 1$
- By previous theorem, there exists s, t such that $1 = s \cdot a + t \cdot b$
- Multiply both sides by c: $c = csa + ctb$
- By earlier theorem, since $a|bc$, $a|ctb$
- Also, by earlier theorem, $a|csa$
- Therefore, $a|csa + ctb$, which implies $a|c$ since $c = csa + ctb$ □

Example

Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.

- Suppose $15 \mid 16 \cdot x$
- Here 15 and 16 are relatively prime
- Thus, previous theorem implies: $15|x$

Question

- Suppose $ca \equiv cb \pmod{m}$. Does this imply $a \equiv b \pmod{m}$?

Another Useful Result

- Theorem: If $ca \equiv cb \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$
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Examples

- If $15x \equiv 15y \pmod{4}$, is $x \equiv y \pmod{4}$?
- If $8x \equiv 8y \pmod{4}$, is $x \equiv y \pmod{4}$?
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Linear Congruences

- A congruence of the form $ax \equiv b \pmod{m}$ where a, b, m are integers and x a variable is called a linear congruence.
- Given such a linear congruence, often need to answer:
  1. Are there any solutions?
  2. What are the solutions?
- Example: Does $8x \equiv 2 \pmod{4}$ have any solutions?
- Example: Does $8x \equiv 2 \pmod{7}$ have any solutions?
- Question: Is there a systematic way to solve linear congruences?

Determining Existence of Solutions

- Theorem: The linear congruence $ax \equiv b \pmod{m}$ has solutions iff $\gcd(a, m) | b$.
- Proof involves two steps:
  1. If $ax \equiv b \pmod{m}$ has solutions, then $\gcd(a, m) | b$.
  2. If $\gcd(a, m) | b$, then $ax \equiv b \pmod{m}$ has solutions.
- First prove (1), then (2).

Proof, Part I

If $ax \equiv b \pmod{m}$ has solutions, then $\gcd(a, m) | b$.

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Proof, Part II

If $\gcd(a, m) | b$, then $ax \equiv b \pmod{m}$ has solutions.

- Let $d = \gcd(a, m)$ and suppose $d | b$
- Then, there is a k such that $b = dk$
- By earlier theorem, there exist s, t such that $d = s \cdot a + t \cdot m$
- Multiply both sides by k: $dk = a \cdot (sk) + m \cdot (tk)$
- Since $b = dk$, we have $b - a \cdot (sk) = m \cdot tk$
- Thus, $b \equiv a \cdot (sk) \pmod{m}$
- Hence, sk is a solution. □

Examples

- Does $5x \equiv 7 \pmod{15}$ have any solutions?
- Does $3x \equiv 4 \pmod 7$ have any solutions?

Finding Solutions

- Can determine existence of solutions, but how to find them?
- Theorem: Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod m$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \ \text{ where } u \in \mathbb{Z}$$

Example

Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod m$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \ \text{ where } u \in \mathbb{Z}$$

- What are the solutions to the linear congruence $3x \equiv 4 \pmod 7$?
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Another Example

Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod m$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \ \text{ where } u \in \mathbb{Z}$$

- What are the solutions to the linear congruence $3x \equiv 1 \pmod 7$?
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Inverse Modulo m

- The inverse of a modulo m, written $\overline{a}$ has the property:

$$a\overline{a} \equiv 1 \pmod m$$

- Theorem: Inverse of a modulo m exists if and only if a and m are relatively prime.
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- Does 3 have an inverse modulo 7?

Example

- Find an inverse of 3 modulo 7.
- An inverse is any solution to $3x \equiv 1 \pmod 7$
- Earlier, we already computed solutions for this equation as:

$$x = -2 + 7u$$

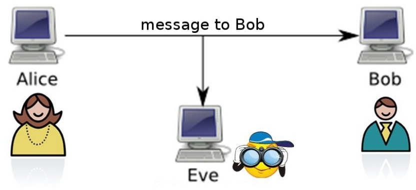
- Thus, -2 is an inverse of 3 modulo 7
- $5, 12, -9, \ldots$ are also inverses

Example 2

- Find inverse of 2 modulo 5.

Cryptography

- Cryptography is the study of techniques for secure transmission of information in the presence of adversaries



- How can Alice send secrete messages to Bob without Eve being able to read them?

Private vs. Public Crypto Systems

- Two different kinds of cryptography systems:
  1. Private key cryptography (also known as symmetric)
  2. Public key cryptography (asymmetric)
- In private key cryptography, sender and receiver agree on secret key that both use to encrypt/decrypt the message
- In public key crytography, a public key is used to encrypt the message, and private key is used to decrypt the message

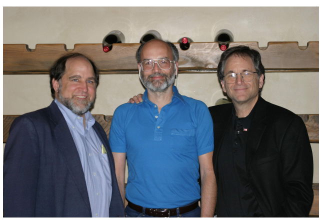
Private Key Cryptography

- Private key crypto is classical method, used since antiquity
- Caesar's cipher is an example of private key cryptography
- Caesar's cipher is shift cipher where $f(p) = (p + k) \pmod{26}$
- Both receiver and sender need to know k to encrypt/decrypt
- Modern symmetric algorithms: RC4, DES, AES, . . .
- Main problem: How do you exchange secret key in a secure way?

Public Key Cryptography

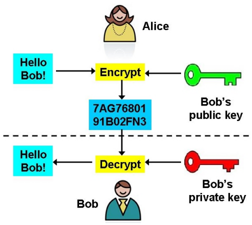
- Public key cryptography is the modern method: different keys are used to encrypt vs. decrypt message
- Most commonly used public key system is RSA
- Great application of number theory and things we've learned

RSA History



- Named after its inventors Rivest, Shamir, and Adlemann, all researchers at MIT (1978)
- Actually, similar system invented earlier by British researcher Clifford Cocks, but classified – unknown until 90's

RSA Overview



- Bob has two keys: public and private
- Everyone knows Bob's public key, but only he knows his private key
- Alice encrypts message using Bob's public key
- Bob decrypts message using private key
- Since public key cannot decrypt, noone can read message accept Bob

High Level Math Behind RSA

- In the RSA system, private key consists of two very large prime numbers p, q
- Public key consists of a number n, which is the product of p, q and another number e, which is relatively prime with $(p-1)(q-1)$
- Encrypt messages using n, e, but to decrypt, must know p, q
- In theory, can extract p, q from n using prime factorization, but this is intractable for very large numbers
- **Security of RSA relies on inherent computational difficulty of prime factorization**

Encryption in RSA

- To send message to Bob, Alice first represents message as a sequence of numbers
- Call this number representing message M
- Alice then uses Bob's public key n, e to perform encryption as:

$$C = M^e \pmod{n}$$

- C is called the ciphertext

RSA Decryption

- Decryption key d is the inverse of e modulo $(p-1)(q-1)$:

$$d \cdot e \equiv 1 \ (\bmod (p-1)(q-1))$$

- Decryption function: $C^d \pmod{n}$
- As we saw earlier, d can be computed reasonably efficiently if we know $(p-1)(q-1)$
- However, since adversaries do not know p, q, they cannot compute d with reasonable computational effort!

Security of RSA

- The encryption function used in RSA is a trapdoor function
- Trapdoor function is easy to compute in one direction, but very difficult in reverse direction without additional knowledge
- Decryption without private key is very hard because requires prime factorization (which is intractable for large enough numbers)
- **Interesting fact:** There are efficient (poly-time) prime factorization algorithms for quantum computers (e.g., Shor's algorithm)
- If we could build quantum computers with sufficient "qubits", RSA would no longer be secure!