CS311H: Discrete Mathematics

More Number Theory

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Review

- What is the fundamental theorem of arithmetic?
- Complete the following:
  - $gcd(a, b) \cdot lcm(a, b) = ?$
  - $gcd(a, b) = gcd(b, ?)$
- Why is the previous equation useful?
- $gcd(a, b)$ can be expressed as ... ?
- What does it mean for a, b to be relatively prime?

A Useful Result

- Suppose $a|bc$. Does that mean $a|c$?
- Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.
- Proof: Since a, b are relatively prime $\gcd(a, b) = 1$
- By previous theorem, there exists s, t such that $1 = s \cdot a + t \cdot b$
- Multiply both sides by c: $c = csa + ctb$
- By earlier theorem, since $a|bc$, $a|ctb$
- Also, by earlier theorem, $a|csa$
- Therefore, $a|csa + ctb$, which implies $a|c$ since $c = csa + ctb$ □

Example

Lemma: If a, b are relatively prime and $a|bc$, then $a|c$.

- Suppose $15 \mid 16 \cdot x$
- Here 15 and 16 are relatively prime
- Thus, previous theorem implies: $15|x$

Question

- Suppose $ca \equiv cb \pmod{m}$. Does this imply $a \equiv b \pmod{m}$?
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Another Useful Result

- Theorem: If $ca \equiv cb \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$
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Examples

- If $15x \equiv 15y \pmod 4$, is $x \equiv y \pmod 4$?
- If $8x \equiv 8y \pmod 4$, is $x \equiv y \pmod 4$?
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Linear Congruences

- A congruence of the form $ax \equiv b \pmod m$ where a, b, m are integers and x a variable is called a linear congruence.
- Given such a linear congruence, often need to answer:
  1. Are there any solutions?
  2. What are the solutions?
- Example: Does $8x \equiv 2 \pmod 4$ have any solutions?
- Example: Does $8x \equiv 2 \pmod 7$ have any solutions?
- Question: Is there a systematic way to solve linear congruences?

Determining Existence of Solutions

- Theorem: The linear congruence $ax \equiv b \pmod m$ has solutions iff $\gcd(a, m) | b$.
- Proof involves two steps:
  1. If $ax \equiv b \pmod m$ has solutions, then $\gcd(a, m) | b$.
  2. If $\gcd(a, m) | b$, then $ax \equiv b \pmod m$ has solutions.
- First prove (1), then (2).

Proof, Part I

If $ax \equiv b \pmod m$ has solutions, then $\gcd(a, m) | b$.

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Proof, Part II

If $\gcd(a, m) | b$, then $ax \equiv b \pmod m$ has solutions.

- Let $d = \gcd(a, m)$ and suppose $d | b$
- Then, there is a k such that $b = dk$
- By earlier theorem, there exist s, t such that $d = s \cdot a + t \cdot m$
- Multiply both sides by k: $dk = a \cdot (sk) + m \cdot (tk)$
- Since $b = dk$, we have $b - a \cdot (sk) = m \cdot tk$
- Thus, $b \equiv a \cdot (sk) \pmod m$
- Hence, sk is a solution. $\square$

Examples

- Does $5x \equiv 7 \pmod{15}$ have any solutions?
- Does $3x \equiv 4 \pmod 7$ have any solutions?

Finding Solutions

- Can determine existence of solutions, but how to find them?
- Theorem: Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod{m}$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

Example

Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod{m}$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

- What are the solutions to the linear congruence $3x \equiv 4 \pmod{7}$?
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Another Example

Let $d = \gcd(a, m) = sa + tm$. If $d|b$, then the solutions to $ax \equiv b \pmod{m}$ are given by:

$$x = \frac{sb}{d} + \frac{m}{d}u \quad \text{where } u \in \mathbb{Z}$$

- What are the solutions to the linear congruence $3x \equiv 1 \pmod{7}$?
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Inverse Modulo m

- The inverse of a modulo m, written $\overline{a}$ has the property:

$$a\overline{a} \equiv 1 \pmod{m}$$

- Theorem: Inverse of a modulo m exists if and only if a and m are relatively prime.
- Proof: Inverse must satisfy $ax \equiv 1 \pmod{m}$
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- Does 3 have an inverse modulo 7?

Example

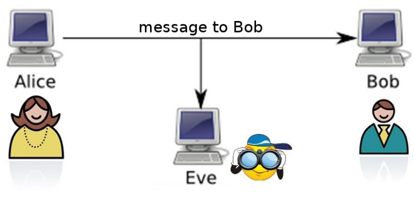
- Find an inverse of 3 modulo 7.
- An inverse is any solution to $3x \equiv 1 \pmod{7}$
- Earlier, we already computed solutions for this equation as:

$$x = -2 + 7u$$

- Thus, -2 is an inverse of 3 modulo 7
- $5, 12, -9, \ldots$ are also inverses

Cryptography

- Cryptography is the study of techniques for secure transmission of information in the presence of adversaries



- How can Alice send secrete messages to Bob without Eve being able to read them?

Private vs. Public Crypto Systems

- Two different kinds of cryptography systems:
  1. Private key cryptography (also known as symmetric)
  2. Public key cryptography (asymmetric)
- In private key cryptography, sender and receiver agree on secret key that both use to encrypt/decrypt the message
- In public key crytography, a public key is used to encrypt the message, and private key is used to decrypt the message

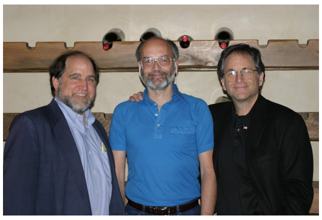
Private Key Cryptography

- Private key crypto is classical method, used since antiquity
- Caesar's cipher is an example of private key cryptography
- Caesar's cipher is shift cipher where $f(p) = (p + k) \pmod{26}$
- Both receiver and sender need to know k to encrypt/decrypt
- Modern symmetric algorithms: RC4, DES, AES, ...
- Main problem: How do you exchange secret key in a secure way?

Public Key Cryptography

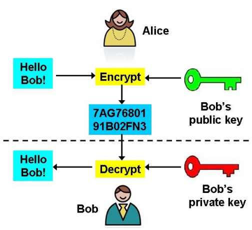
- Public key cryptography is the modern method: different keys are used to encrypt vs. decrypt message
- Most commonly used public key system is RSA
- Great application of number theory and things we've learned

RSA History



- Named after its inventors Rivest, Shamir, and Adlemann, all researchers at MIT (1978)
- Actually, similar system invented earlier by British researcher Clifford Cocks, but classified – unknown until 90's

RSA Overview



- Bob has two keys: public and private
- Everyone knows Bob's public key, but only he knows his private key
- Alice encrypts message using Bob's public key
- Bob decrypts message using private key
- Since public key cannot decrypt, noone can read message accept Bob

High Level Math Behind RSA

- In the RSA system, private key consists of two very large prime numbers p, q
- Public key consists of a number n, which is the product of p, q and another number e, which is relatively prime with $(p-1)(q-1)$
- Encrypt messages using n, e, but to decrypt, must know p, q
- In theory, can extract p, q from n using prime factorization, but this is intractable for very large numbers
- **Security of RSA relies on inherent computational difficulty of prime factorization**

Encryption in RSA

- To send message to Bob, Alice first represents message as a sequence of numbers
- Call this number representing message M
- Alice then uses Bob's public key n, e to perform encryption as:

$$C = M^e \pmod{n}$$

- C is called the ciphertext

RSA Decryption

- Decryption key d is the inverse of e modulo $(p-1)(q-1)$:

$$d \cdot e \equiv 1 \pmod{(p-1)(q-1)}$$

- Decryption function: $C^d \pmod{n}$
- As we saw earlier, d can be computed reasonably efficiently if we know $(p-1)(q-1)$
- However, since adversaries do not know p, q, they cannot compute d with reasonable computational effort!

Security of RSA

- The encryption function used in RSA is a trapdoor function
- Trapdoor function is easy to compute in one direction, but very difficult in reverse direction without additional knowledge
- Decryption without private key is very hard because requires prime factorization (which is intractable for large enough numbers)
- **Interesting fact:** There are efficient (poly-time) prime factorization algorithms for quantum computers (e.g., Shor's algorithm)
- If we could build quantum computers with sufficient "qubits", RSA would no longer be secure!