Sets and Basic Concepts

- A set is an unordered collection of distinct objects
- Example: Positive even numbers less than 10: \{2, 4, 6, 8\}
- Objects in set \( S \) are called members (or elements) of that set
- If \( x \) is a member of \( S \), we write \( x \in S \)
- \# elements in a set is called its cardinality, written \(|S|\)

Important Sets in Mathematics

- Many sets that play fundamental role in mathematics have infinite cardinality
- Set of integers \( \mathbb{Z} = \{-\ldots, -2, -1, 0, 1, 2, \ldots\} \)
- Set of positive integers: \( \mathbb{Z}^+ = \{1, 2, \ldots\} \)
- Natural numbers: \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \)
- Set of real numbers: \( \mathbb{R} = \{\pi, -\ldots, -1.999, \ldots, 0, \ldots, 0.000001, \ldots\} \)

Special Sets

- The universal set, written \( U \), includes all objects under consideration
- The empty set, written \( \emptyset \) or \( \{\} \), contains no objects
- A set containing exactly one element is called a singleton set
- What special set is \( S = \{x \mid p(x) \land \lnot p(x)\} \) equal to?
- What special set is \( S = \{x \mid p(x) \lor \lnot p(x)\} \) equal to?

Set Builder Notation

- Infinite sets are often written using set builder notation
  \[ S = \{x \mid x \text{ has property } p\} \]
- Example: \( S = \{x \mid x \in \mathbb{Z} \land x\%2 = 0\} \)
- Which set is \( S \)?
- Example: \( \mathbb{Q} = \{p/q \mid p \in \mathbb{Z} \land q \in \mathbb{Z} \land q \neq 0\} \)
- Which set if \( \mathbb{Q} \)?

Subsets and Supersets

- A set \( A \) is a subset of set \( B \), written \( A \subseteq B \), iff every element in \( A \) is also an element of \( B \) \((\forall x. x \in A \Rightarrow x \in B)\)
  \[ \subseteq \]
  \[ A \subseteq B \]
- If \( A \subseteq B \), then \( B \) is called a superset of \( A \), written \( B \supseteq A \)
- A set \( A \) is a proper subset of set \( B \), written \( A \subset B \), iff:
  \((\forall x. x \in A \Rightarrow x \in B) \land (\exists x. x \in B \land x \notin A)\)
- Sets \( A \) and \( B \) are equal, written \( A = B \), if \( A \subseteq B \) and \( B \subseteq A \)
Power Set

- The **power set** of a set $S$, written $P(S)$, is the set of all subsets of $S$.
- **Example:** What is the powerset of $\{a, b, c\}$?
- **Fact:** If cardinality of $S$ is $n$, then $|P(S)| = 2^n$
- **What is the power set of $\emptyset$?**
- **What is the power set of $\{\emptyset\}$?**

Ordered Tuples

- **An important operation on sets is called Cartesian product**
- To define Cartesian product, need ordered tuples
- An ordered $n$-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection with $a_1$ as its first element, $a_2$ as its second element, ..., and $a_n$ as its last element.
- **Observe:** $(1, 2)$ and $(2, 1)$ are not the same!
- Tuple of two elements called **pair** (3 elements called **triple**)

Cartesian Product

- The **Cartesian product** of two sets $A$ and $B$, written $A \times B$, is the set of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$
  \[ A \times B = \{(a, b) \mid a \in A \land b \in B\} \]
- **Example:** Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. What is $A \times B$?
- **Example:** What is $B \times A$?
- **Observe:** $A \times B \neq B \times A$ in general!
- **Observe:** If $|A| = n$ and $|B| = m$, $|A \times B|$ is $nm$.

More on Cartesian Products

- Cartesian product generalizes to more than two sets
- Cartesian product of $A_1 \times A_2 \ldots \times A_n$ is the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$
- **Example:** If $A = \{1, 2\}, B = \{a, b\}, C = \{*, \circ\}$, what is $A \times B \times C$?

Set Operations

- Set union:
  \[ A \cup B = \{x \mid x \in A \lor x \in B\} \]
- Intersection:
  \[ A \cap B = \{x \mid x \in A \land x \in B\} \]
- Difference:
  \[ A - B = \{x \mid x \in A \land x \notin B\} \]
- Complement:
  \[ \overline{A} = \{x \mid x \notin A\} \]

Disjoint Sets

- Two set $A$ and $B$ are called **disjoint** if $A \cap B = \emptyset$
Exercise

Prove De Morgan’s law for sets: \( A \cup B = A \cap B \)

Proving Distributivity of \( \cap \)

▶ Prove \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)

Naive Set Theory and Russell’s Paradox

▶ Intuitive formulation of sets is called naive set theory – goes back to German mathematician George Cantor (1800’s)
▶ In naive set theory, any definable collection is a set (axiom of unrestricted comprehension)
▶ In other words, unrestricted comprehension says that \( \{x \mid P(x)\} \) is a set, for any property \( P \)
▶ In 1901, Bertrand Russell showed that Cantor’s set theory is inconsistent
▶ This can be shown using so-called Russell’s paradox

Russell’s Paradox

▶ Let \( R \) be the set of sets that are not members of themselves:
\[
R = \{ S \mid S \not\in S \}
\]
▶ Two possibilities: Either \( R \in R \) or \( R \not\in R \)
▶ Suppose \( R \in R \).
▶ But by definition of \( R \), \( R \) does not have itself as a member, i.e., \( R \not\in R \)
▶ But this contradicts \( R \in R \)

Russell’s Paradox, cont.

▶ Now suppose \( R \not\in R \) (i.e., \( R \) not a member of itself)
▶ But since \( R \) is the set of sets that are not members of themselves, \( R \) must be a member of \( R \)!
▶ This shows that set \( R \) cannot exist, contradicting the axiom of unrestricted comprehension!!
▶ Since we have a contradiction, one can prove any nonsense using naive set theory!
▶ Much research on consistent versions of set theory ⇒ Zermelo’s ZFC, Russell’s type theory etc.

Illustration of Russell’s Paradox

▶ Russell’s paradox and other similar paradoxes inspired artists at the turn of the century, esp. Escher and Magritte
▶ Belgian painter Rene Magritte made a graphical illustration of Russell’s paradox:
Undecidability

- A proof similar to Russell’s paradox can be used to show undecidability of the famous Halting problem
- A decision problem is a question of a formal system that has a yes or no answer
- Example: satisfiability/valid in FOL or propositional logic
- A decision problem is undecidable if it is not possible to have an algorithm that always terminates and gives correct answer

The Halting Problem

- The famous Halting problem in CS undecidable.
- Halting problem: Given a program $P'$ and an input $w$, does $P'$ terminate on $w$?
- What does it mean for this problem to be (un)decidable?

Proof of Undecidability of Halting Problem

- Assume such a program $P$ exists
- Now, construct program $P'$ such that $P'$ halts iff its input does not halt on itself:

Other Famous Undecidable Problems

- Validity in first-order logic: Given an arbitrary first order logic formula $F$, is $F$ valid? (Hilbert’s Entscheidungsproblem)
- Program verification: Given a program $P$ and a non-trivial property $Q$, does $P$ satisfy property $Q$? (Rice’s theorem)
- Hilbert’s 10th problem: Does a diophantine equation $p(x_1, \ldots, x_n) = 0$ have solutions? (i.e., integer solutions)
Provability and Computability

- If paradoxes and computability/provability proofs interest you...
  - Take theory of computation and mathematical logic courses
  - Book recommendation: “Godel, Escher, Bach” by Douglas Hofstadter

Exercise: Barber’s paradox

- According to an ancient Sicilian legend, a remote town can only be reached by traveling a dangerous mountain road.
- The barber of this town shaves all those people, and only those people, who do not shave themselves.
- Can such a barber exist?