Converse of a Implication

- Recall implication \( p \rightarrow q \) – when does it evaluate to false?
- The converse of an implication \( p \rightarrow q \) is \( q \rightarrow p \).
- What is the converse of "If I am a CS major, then I can program"?
- What is the converse of "If I get an A in CS311, then I am smart"?
- Is it possible for a implication to be true, but its converse to be false?

Inverse of an Implication

- The inverse of an implication \( p \rightarrow q \) is \( \neg p \rightarrow \neg q \).
- What is the inverse of "If I am a CS major, then I can program"?
- What is the inverse of "If I get an A in CS311, then I am smart"?
- Is it possible for a implication to be true, but its inverse to be false?

Contrapositive of Implication

- The contrapositive of an implication \( p \rightarrow q \) is \( \neg q \rightarrow \neg p \).
- What is the contrapositive of "If I am a CS major, then I can program"?
- What is the contrapositive of "If I get an A in CS311, then I am smart"?
- Question: Is it possible for an implication to be true, but its contrapositive to be false?

Conditional and its Contrapositive

A conditional \( p \rightarrow q \) and its contrapositive \( \neg q \rightarrow \neg p \) always have the same truth value.
- Prove it!

Question

- Consider a conditional \( p \rightarrow q \)
- Is it possible that its converse is true, but inverse is false?
Summary

- Conditional is of the form $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$
- Contrapositive: $\neg q \rightarrow \neg p$
- Conditional and contrapositive have same truth value
- Inverse and converse always have same truth value

Biconditionals

- A biconditional $p \leftrightarrow q$ is the proposition "$p$ if and only if $q$".
- The biconditional $p \leftrightarrow q$ is true if $p$ and $q$ have same truth value, and false otherwise.
- Exercise: Construct a truth table for $p \leftrightarrow q$
- Question: How can we express $p \leftrightarrow q$ using the other boolean connectives?

Operator Precedence

- Given a formula $p \land q \lor r$, do we parse this as $(p \land q) \lor r$ or $p \land (q \lor r)$?
- Without settling on a convention, formulas without explicit paranthesization are ambiguous.
- To avoid ambiguity, we will specify precedence for logical connectives.

Operator Precedence, cont.

- Negation ($\neg$) has higher precedence than all other connectives.
- Question: Does $\neg p \land q$ mean (i) $\neg(p \land q)$ or (ii) $(\neg p) \land q$?
- Conjunction ($\land$) has next highest predence.
- Question: Does $p \land q \lor q$ mean (i) $(p \land q) \lor r$ or (ii) $p \land (q \lor r)$?
- Disjunction ($\lor$) has third highest precedence.
- Next highest is precedence is $\rightarrow$, and lowest precedence is $\leftrightarrow$

Operator Precedence Example

- Which is the correct interpretation of the formula $p \lor q \land r \leftrightarrow q \rightarrow \neg r$?
  
  (A) $(p \lor (q \land r)) \leftrightarrow q \rightarrow \neg r$
  (B) $(p \lor q) \land r \leftrightarrow q \rightarrow \neg r$
  (C) $(p \lor (q \land r)) \leftrightarrow (q \rightarrow \neg r)$
  (D) $(p \lor ((q \land r) \leftrightarrow q)) \rightarrow \neg r$

Validity, Unsatisfiability

- In general, truth value of a propositional formula depends on truth assignments to variables
  
  - But some formulas evaluate to true for every assignment, e.g., $p \lor \neg p$
  - Such formulas are called tautologies or valid formulas
  - Some formulas evaluate to false for every assignment, e.g., $p \land \neg p$
  - Such formulas are called unsatisfiable formulas or contradictions
Interpretations

- To make satisfiability/validity precise, we’ll define interpretation of formula.
- An interpretation $I$ for a formula $F$ is a mapping from each propositional variables in $F$ to exactly one truth value $I: \{ p \mapsto \text{true}, q \mapsto \text{false}, \ldots \}$.
- Each interpretation corresponds to one row in the truth table, so $2^n$ possible interpretations.

Entailment

- Under an interpretation, every propositional formula evaluates to $T$ or $F$.
- Formula $F$ + Interpretation $I = \text{Truth value}$.
- We write $I \models F$ if $F$ evaluates to true under $I$.
- Similarly, $I \nvDash F$ if $F$ evaluates to false under $I$.
- Theorem: $I \models F$ if and only if $I \nvDash \neg F$.

Examples

- Consider the formula $F: p \land q \rightarrow \neg p \lor \neg q$.
- Let $I_1$ be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{false}]$.
- What does $F$ evaluate to under $I_1$?
- Thus, $I_1 \models F$.
- Let $I_2$ be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$.
- What does $F$ evaluate to under $I_2$?
- Thus, $I_2 \nvDash F$.

Another Example

- Let $F_1$ and $F_2$ be two propositional formulas.
- Suppose $F_1$ evaluates to true under interpretation $I$.
- What does $F_2 \land \neg F_1$ evaluate to under $I$?

Satisfiability, Validity

- $F$ is satisfiable iff there exists interpretation $I$ s.t. $I \models F$.
- $F$ is valid iff for all interpretations $I$, $I \models F$.
- $F$ is unsatisfiable iff for all interpretations $I$, $I \nvDash F$.
- $F$ is contingent if it is satisfiable, but not valid.

True/False Questions

- Are the following statements true or false?
  - If a formula is valid, then it is also satisfiable.
  - If a formula is satisfiable, then its negation is unsatisfiable.
  - If $F_1$ and $F_2$ are satisfiable, then $F_1 \land F_2$ is also satisfiable.
  - If $F_1$ and $F_2$ are satisfiable, then $F_1 \lor F_2$ is also satisfiable.
Duality Between Validity and Unsatisfiability

*Proof:*

\[ F \text{ is valid if and only if } \neg F \text{ is unsatisfiable} \]

Proving Validity

*Question:* How can we prove that a propositional formula is a tautology?

*Exercise:* Which formulas are tautologies? Prove your answer.

1. \((p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)\)
2. \((p \land q) \lor \neg p\)

Proving Satisfiability, Unsatisfiability, Contingency

*Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:*

- **Satisfiable:** There exists a row where formula evaluates to true
- **Unsatisfiable:** In all rows, formula evaluates to false
- **Contingent:** Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercise

*Is \((p \rightarrow q) \rightarrow (q \rightarrow p)\) valid, unsatisfiable, or contingent? Prove your answer.*

Implication

- **Formula \(F_1\) implies \(F_2\) (written \(F_1 \Rightarrow F_2\)) iff for all interpretations \(I, I \models F_1 \rightarrow F_2\)**

\[ F_1 \Rightarrow F_2 \text{ iff } F_1 \rightarrow F_2 \text{ is valid} \]

- **Caveat:** \(F_1 \Rightarrow F_2\) is not a propositional logic formula; \(\Rightarrow\) is not part of PL syntax!
- **Instead, \(F_1 \Rightarrow F_2\) is a semantic judgment, like satisfiability!**

Example

*Does \(p \lor q\) imply \(p\)? Prove your answer.*
Equivalence

- Two formulas $F_1$ and $F_2$ are equivalent if they have the same truth value for every interpretation, e.g., $p \lor p$ and $p$

- More precisely, formulas $F_1$ and $F_2$ are equivalent, written $F_1 \equiv F_2$ or $F_1 \iff F_2$, iff:

$$F_1 \iff F_2 \iff F_1 \leftrightarrow F_2 \text{ is valid}$$

- $\iff$, $\leftrightarrow$ not part of PL syntax; they are semantic judgments!

Important Equivalences

- Some important equivalences are useful to know!

  - Law of double negation: $\neg\neg\phi \equiv \phi$

  - Identity Laws: $\phi \land T \equiv \phi \quad \phi \lor F \equiv \phi$

  - Domination Laws: $\phi \lor T \equiv T \quad \phi \land F \equiv \phi$

  - Idempotent Laws: $\phi \lor \phi \equiv \phi \quad \phi \land \phi \equiv \phi$

  - Negation Laws: $\phi \land \neg\phi \equiv F \quad \phi \lor \neg\phi \equiv T$

  - Absorption Laws: $\phi_1 \land (\phi_1 \lor \phi_2) \equiv \phi_1 \quad \phi_1 \lor (\phi_1 \land \phi_2) \equiv \phi_2$

De Morgan’s Laws

- Let $\text{cs311}$ be the proposition “John took CS311” and $\text{cs314}$ be the proposition “John took CS314”

- In simple English what does $\neg (\text{cs311} \land \text{cs314})$ mean?

- DeMorgan’s law expresses exactly this equivalence!

  - De Morgan’s Law #1: $\neg(p \land q) \equiv (\neg p \lor \neg q)$

  - De Morgan’s Law #2: $\neg(p \lor q) \equiv (\neg p \land \neg q)$

- When you “push” negations in, $\land$ becomes $\lor$ and vice versa

Example

- Prove that $p \rightarrow q$ and $\neg p \lor q$ are equivalent

Why are These Equivalences Useful?

- Use known equivalences to prove that two formulas are equivalent

  - i.e., rewrite one formula into another using known equivalences

  - Examples: Prove following formulas are equivalent:

    1. $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$
    2. $\neg(p \rightarrow q)$ and $p \land \neg q$
Formalizing English Arguments in Logic

We can use logic to prove correctness of English arguments.

For example, consider the argument:

- If Joe drives fast, he gets a speeding ticket.
- Joe did not get a ticket.
- Therefore, Joe did not drive fast.

Let \( f \) be the proposition "Joe drives fast", and \( t \) be the proposition "Joe gets a ticket".

How do we encode this argument as a logical formula?

Example, cont

"If Joe drives fast, he gets a speeding ticket. Joe did not get a ticket. Therefore, he did not drive fast."

\((f \rightarrow t) \land \neg t \rightarrow \neg f\)

How can we prove this argument is valid?

Can do this in two ways:

1. Use truth table to show formula is tautology
2. Use known equivalences to rewrite formula to true

Another Example

Can also use logic to prove an argument is not valid.

Suppose your friend George make the following argument:

- If Jill carries an umbrella, it is raining.
- Jill is not carrying an umbrella.
- Therefore it is not raining.

Let \( u = " \text{Jill is carrying an umbrella}" \), and \( r = " \text{it is raining}" \)

How do we encode this argument in logic?

Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining."

\((u \rightarrow r) \land \neg u \rightarrow \neg r\)

How can we prove George’s argument is invalid?

Summary

- A formula is valid if it is true for all interpretations.
- A formula is satisfiable if it is true for at least one interpretation.
- A formula is unsatisfiable if it is false for all interpretations.
- A formula is contingent if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas \( F_1 \) and \( F_2 \) are equivalent, written \( F_1 \equiv F_2 \), if \( F_1 \leftrightarrow F_2 \) is valid.