Announcements

- First homework assignment out today!
- Due in one week, i.e., before lecture next Tuesday

Converse of a Implication

- The converse of an implication $p \rightarrow q$ is $q \rightarrow p$.
- What is the converse of "If I am a CS major, then I can program"?
- What is the converse of "If I get an A in CS311, then I am smart"?
- Question: Do an implication and its converse always have same truth value?

Inverse of an Implication

- The inverse of an implication $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- What is the inverse of "If I am a CS major, then I can program"?
- What is the inverse of "If I get an A in CS311, then I am smart"?
- Question: Do an implication and its converse always have same truth value?

Contrapositive of Implication

- The contrapositive of an implication $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
- What is the contrapositive of "If I am a CS major, then I can program"?
- What is the contrapositive of "If I get an A in CS311, then I am smart"?
- Very important: An implication and its contrapositive always have the same truth value

Conditional and its Contrapositive

Prove that a conditional $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ always have the same truth value.
Question

- Given a conditional $p \rightarrow q$, is it possible that its converse is true, but inverse is false? Prove it!

Summary

- Conditional is of the form $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$
- Contrapositive: $\neg q \rightarrow \neg p$
- Conditional and contrapositive have same truth value
- Inverse and converse always have same truth value

Biconditionals

- A biconditional $p \leftrightarrow q$ is the proposition "$p$ if and only if $q$".
- The biconditional $p \leftrightarrow q$ is true if $p$ and $q$ have same truth value, and false otherwise.
- Exercise: Construct a truth table for $p \leftrightarrow q$
- Question: How can we express $p \leftrightarrow q$ using the other boolean connectives?

Operator Precedence

- Negation ($\neg$) has higher precedence than all other connectives.
- Question: Does $\neg p \land q$ mean (i) $\neg(p \land q)$ or (ii) $(\neg p) \land q$?
- Conjunction ($\land$) has next highest precedence.
- Question: Does $p \land q \lor q$ mean (i) $(p \land q) \lor q$ or (ii) $p \land (q \lor r)$?
- Disjunction ($\lor$) has third highest precedence.
- Next highest is precedence is $\rightarrow$, and lowest precedence is $\leftrightarrow$

Operator Precedence Example

- Which is the correct interpretation of the formula $p \lor q \land r \leftrightarrow q \rightarrow \neg r$

(A) $(p \lor (q \land r)) \leftrightarrow q \rightarrow (\neg r)$
(B) $(p \lor q) \land r \leftrightarrow q \rightarrow (\neg r)$
(C) $(p \lor (q \land r)) \leftrightarrow (q \rightarrow (\neg r))$
(D) $(p \lor ((q \land r) \leftrightarrow q)) \rightarrow (\neg r)$
Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- Example: \( \neg p \) evaluates to true under assignment \( p = F \) and to false under \( p = T \)
- Some formulas evaluate to true for every assignment, e.g., \( p \lor \neg p \)
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., \( p \land \neg p \)
- Such formulas are called unsatisfiable formulas or contradictions

Interpretations

- To make satisfiability/validity precise, we’ll define interpretation of formula
- An interpretation \( I \) for a formula \( F \) is a mapping from each propositional variable in \( F \) to exactly one truth value, e.g., \( I: \{ p \mapsto \text{true}, q \mapsto \text{false}, \ldots \} \)
- Each interpretation corresponds to one row in the truth table, so \( 2^n \) possible interpretations

Entailment

- Under an interpretation, every propositional formula evaluates to T or F
- Formula \( F \) + Interpretation \( I = \text{Truth value} \)
- We write \( I \models F \) if \( F \) evaluates to true under \( I \)
- Similarly, \( I \not\models F \) if \( F \) evaluates to false under \( I \).
- Observe: \( I \models F \) if and only if \( I \not\models \neg F \)

Examples

- Consider the formula \( F: p \land q \rightarrow \neg p \lor \neg q \)
- Let \( I_1 \) be the interpretation such that \( [p \mapsto \text{true}, q \mapsto \text{false}] \)
- What does \( F \) evaluate to under \( I_1 \)?
- Thus, \( I_1 \models F \)
- Let \( I_2 \) be the interpretation such that \( [p \mapsto \text{true}, q \mapsto \text{true}] \)
- What does \( F \) evaluate to under \( I_2 \)?
- Thus, \( I_2 \not\models F \)

Another Example

- Let \( F_1 \) and \( F_2 \) be two propositional formulas
- Suppose \( F_1 \) evaluates to true under interpretation \( I \)
- What does \( F_2 \land \neg F_1 \) evaluate to under \( I \)?

Satisfiability, Validity

- \( F \) is satisfiable iff there exists interpretation \( I \) s.t. \( I \models F \)
- \( F \) is valid iff for all interpretations \( I \), \( I \models F \)
- \( F \) is unsatisfiable iff for all interpretations \( I \), \( I \not\models F \)
- \( F \) is contingent if it is satisfiable, but not valid.
True/False Questions

Are the following statements true or false?

- If a formula is valid, then it is also satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable.
- If $F_1$ and $F_2$ are satisfiable, then $F_1 \land F_2$ is also satisfiable.
- If $F_1$ and $F_2$ are satisfiable, then $F_1 \lor F_2$ is also satisfiable.

Duality Between Validity and Unsatisfiability

$F$ is valid if and only if $\neg F$ is unsatisfiable.

- Proof:

Proving Validity

- **Question:** How can we prove that a propositional formula is a tautology?
- **Exercise:** Which formulas are tautologies? Prove your answer.
  1. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
  2. $(p \land q) \lor \neg p$

Proving Satisfiability, Unsatisfiability, Contingency

- Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:
  - **Satisfiable:** There exists a row where formula evaluates to true
  - **Unsatisfiable:** In all rows, formula evaluates to false
  - **Contingent:** Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercise

- Is $(p \rightarrow q) \rightarrow (q \rightarrow p)$ valid, unsatisfiable, or contingent? Prove your answer.

Implication

- **Formula $F_1$ implies $F_2$** (written $F_1 \Rightarrow F_2$) iff for all interpretations $I$, $I \models F_1 \rightarrow F_2$

  $F_1 \Rightarrow F_2$ iff $F_1 \rightarrow F_2$ is valid

- **Caveat:** $F_1 \Rightarrow F_2$ is not a propositional logic formula; $\Rightarrow$ is not part of PL syntax!
Example

- Does \( p \lor q \) imply \( p \)? Prove your answer.

Equivalence

- Two formulas \( F_1 \) and \( F_2 \) are equivalent if they have same truth value for every interpretation, e.g., \( p \lor p \) and \( p \)

Example

- Prove that \( p \to q \) and \( \neg p \lor q \) are equivalent

Important Equivalences

- Some important equivalences are useful to know!

Commutativity and Distributivity Laws

- Commutative Laws: \( p \lor q \equiv q \lor p \) \( p \land q \equiv q \land p \)

De Morgan’s Laws

- Let \( cs311 \) be the proposition “John took CS311” and \( cs312 \) be the proposition “John took CS312”

- In simple English what does \( \neg(cs311 \land cs312) \) mean?

- DeMorgan’s law expresses exactly this equivalence!

- De Morgan’s Law #1: \( \neg(p \land q) \equiv (\neg p \lor \neg q) \)

- De Morgan’s Law #2: \( \neg(p \lor q) \equiv (\neg p \land \neg q) \)

- When you “push” negations in, \( \land \) becomes \( \lor \) and vice versa
Why are These Equivalences Useful?

- Use known equivalences to prove that two formulas are equivalent
- Examples: Prove following formulas are equivalent:
  1. \( \neg(p \lor (\neg p \land q)) \) and \( \neg p \land \neg q \)
  2. \( \neg(p \rightarrow q) \) and \( p \land \neg q \)

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Formalizing English Arguments in Logic

- We can use logic to prove/disprove arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let \( f \) be the proposition "Joe drives fast", and \( t \) be the proposition "Joe gets a ticket"
- How do you prove using logic that above argument is valid?

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Another Example

- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Is this argument valid? Prove/disprove using logic!
- Let \( u = "Jill is carrying an umbrella" \), and \( r = "It is raining" \)
- How do we encode this argument in logic?

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Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining." \( ((u \rightarrow r) \land \neg u) \rightarrow \neg r \)
- How can we prove George’s argument is invalid?

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Summary

- A formula is valid if it is true for all interpretations.
- A formula is satisfiable if it is true for at least one interpretation.
- A formula is unsatisfiable if it is false for all interpretations.
- A formula is contingent if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas \( F_1 \) and \( F_2 \) are equivalent, written \( F_1 \equiv F_2 \), if \( F_1 \leftrightarrow F_2 \) is valid

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