Why First-Order Logic?

- So far, we studied the simplest logic: propositional logic
- But for some applications, propositional logic is not expressive enough
- First-order logic is more expressive: allows representing more complex facts and making more sophisticated inferences

A Motivating Example

- For instance, consider the statement "Anyone who drives fast gets a speeding ticket"
- From this, we should be able to conclude "If Joe drives fast, he will get a speeding ticket"
- Similarly, we should be able to conclude "If Rachel drives fast, she will get a speeding ticket" and so on.
- But Propositional Logic does not allow inferences like that because we cannot talk about concepts like "everyone", "someone" etc.
- First-order logic (predicate logic) allows making such kinds of inferences

Building Blocks of First-Order Logic

- The building blocks of propositional logic were propositions
- In first-order logic, there are three kinds of basic building blocks: constants, variables, predicates
- Constants: refer to specific objects (in a universe of discourse)
- Examples: George, 6, Austin, CS311, ...
- Variables: range over objects (in a universe of discourse)
- Examples: x,y,z, ...
- If universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio, ...

Building Blocks of First-Order Logic, cont.

- Predicates describe properties of objects or relationships between objects
- Examples: ishappy, betterthan, loves, > ...
- Predicates can be applied to both constants and variables
- Examples: ishappy(George), betterthan(x,y), loves(George, Rachel), x \( > 3 \), ...
- A predicate \( P(x) \) is true or false depending on whether property \( P \) holds for \( x \)
- Example: ishappy(George) is true if George is happy, but false otherwise

Predicate Examples

- Consider predicate \( Q(x,y) \) which indicates that \( x = y + 3 \)
- What is the truth value of \( Q(3,0) \)?
- What is the truth value of \( Q(1,2) \)?
Formulas in First Order Logic

- Formulas in first-order logic are formed using predicates and logical connectives.
- Example: `even(x) ∨ odd(x)` is a formula
- Example: `(odd(x) → ¬ even(x)) ∧ even(x)`

Semantics of First-Order Logic

- In propositional logic, the truth value of a formula depends on a truth assignment to variables.
- In FOL, truth value of a formula depends on interpretation of predicate symbols and variables over some domain \( D \)
- Consider a FOL formula \( \neg P(x) \)
- A possible interpretation:
  \[
  D = \{1, 0\}, \quad P(x) = true, \quad P(0) = false, \quad x = 0
  \]
- Under this interpretation, what’s truth value of \( \neg P(x) \)?
- What about if \( x = 0 \)?

More Examples

- Consider interpretation \( I \) over domain \( D = \{1, 2\} \)
  - \( P(1, 1) = P(1, 2) = true, \quad P(2, 1) = P(2, 2) = false \)
  - \( Q(1) = false, \quad Q(2) = true \)
  - \( x = 1, \ y = 2 \)
- What is truth value of \( P(x, y) ∧ Q(y) \) under \( I \)?
- What is truth value of \( P(y, x) → Q(y) \) under \( I \)?
- What is truth value of \( P(x, y) → Q(x) \) under \( I \)?

Quantifiers

- Real power of first-order logic over propositional logic: quantifiers
- Quantifiers allow us to talk about all objects or the existence of some object
- There are two quantifiers in first-order logic:
  1. Universal quantifier (\( ∀ \)): refers to all objects
  2. Existential quantifier (\( ∃ \)): refers to some object

Universal Quantifiers

- Universal quantification of \( P(x) \), \( ∀x.P(x) \), is the statement “\( P(x) \) holds for all objects \( x \) in the universe of discourse.”
- \( ∀x.P(x) \) is true if predicate \( P \) is true for every object in the universe of discourse, and false otherwise
- Consider domain \( D = \{0, *\} \), \( P(0) = true, \quad P(*) = false \)
- What is truth value of \( ∀x.P(x) \)?
- Object \( o \) for which \( P(o) \) is false is counterexample of \( ∀x.P(x) \)
- What is a counterexample for \( ∀x.P(x) \) in previous example?

More Universal Quantifier Examples

- Consider the domain \( D \) of real numbers and predicate \( P(x) \) with interpretation \( x^2 ≥ x \)
- What is the truth value of \( ∀x.P(x) \)?
- What is a counterexample?
- What if the domain is integers?
- Observe: Truth value of a formula depends on a universe of discourse!
**Existential Quantifiers**

- **Existential quantification** of $P(x)$, written $\exists x. P(x)$, is "There exists an element $x$ in the domain such that $P(x)$".
- $\exists x. P(x)$ is true if there is at least one element in the domain such that $P(x)$ is true.
- In first-order logic, domain is required to be non-empty.
- Consider domain $D = \{0, \star\}$, $P(0) = \text{true}$, $P(\star) = \text{false}$.
- What is truth value of $\exists x. P(x)$?

**Quantified Formulas**

- So far, only discussed how to quantify individual predicates.
- But we can also quantify entire formulas containing multiple predicates and logical connectives.
- $\exists x. (\text{even}(x) \land \text{gt}(x, 100))$ is a valid formula in FOL.
- What's truth value of this formula if domain is all integers?
  - assuming $\text{even}(x)$ means "$x$ is even" and $\text{gt}(x, y)$ means $x > y$.
- What about $\forall x. (\text{even}(x) \lor \text{gt}(x, 100))$?

**Existential Quantifier Examples**

- Consider the domain of reals and predicate $P(x)$ with interpretation $x < 0$.
- What is truth value of $\exists x. P(x)$?
- What if domain is positive integers?
- Let $Q(y)$ be the statement $y > y^2$.
- What is the truth value of $\exists y. Q(y)$ if domain is reals?
- What about if domain is integers?

**More Examples of Quantified Formulas**

- Consider the domain of integers and predicates $\text{even}(x)$ and $\text{div}_4(x)$ which represents if $x$ is divisible by 4.
- What is the truth value of the following quantified formulas?
  - $\forall x. (\text{div}_4(x) \rightarrow \text{even}(x))$
  - $\forall x. (\text{even}(x) \rightarrow \text{div}_4(x))$
  - $\exists x. (\neg\text{div}_4(x) \land \text{even}(x))$
  - $\exists x. (\neg\text{div}_4(x) \rightarrow \text{even}(x))$
  - $\forall x. (\neg\text{div}_4(x) \rightarrow \text{even}(x))$

**Translating English Into Quantified Formulas**

Assuming $\text{freshman}(x)$ means "$x$ is a freshman" and $\text{inCS311}(x)$ "$x$ is taking CS311", express the following in FOL:
- Someone in CS311 is a freshman.
- No one in CS311 is a freshman.
- Everyone taking CS311 are freshmen.
- Every freshman is taking CS311.

**DeMorgan’s Laws for Quantifiers**

- Learned about DeMorgan’s laws for propositional logic:
  
  \[
  \neg(p \land q) \equiv \neg p \lor \neg q \\
  \neg(p \lor q) \equiv \neg p \land \neg q
  \]

- DeMorgan’s laws extend to first-order logic, e.g.,
  
  \[
  \neg(\text{even}(x) \lor \text{div}_4(x)) \equiv (\neg\text{even}(x) \land \neg\text{div}_4(x))
  \]

- Two new DeMorgan’s laws for quantifiers:
  
  \[
  \neg\forall x. P(x) \equiv \exists x. \neg P(x) \\
  \neg\exists x. P(x) \equiv \forall x. \neg P(x)
  \]

- When you push negation in, $\forall$ flips to $\exists$ and vice versa.
Using DeMorgan’s Laws

- Expressed “No one in CS311 is a freshman” as 
  \[ \neg \exists x. (\text{inCS311}(x) \land \text{freshman}(x)) \]
- Let’s apply DeMorgan’s law to this formula:
  \[ \neg (\exists x. (\text{inCS311}(x) \land \text{freshman}(x))) \]
- Using the fact that \( p \rightarrow q \) is equivalent to \( \neg p \lor q \), we can write this formula as:
  \[ \neg \neg \exists x. (\text{inCS311}(x) \land \neg \text{freshman}(x)) \]
- Therefore, these two formulas are equivalent!

Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers
- For example, can’t express “Everybody loves someone” using a single quantifier
- Suppose predicate \( \text{loves}(x, y) \) means “Person \( x \) loves person \( y \)”
- What does \( \forall x. \exists y. \text{loves}(x, y) \) mean?
- What does \( \exists y. \forall x. \text{loves}(x, y) \) mean?
- Observe: Order of quantifiers is very important!

More Nested Quantifier Examples

Using the \( \text{loves}(x, y) \) predicate, how can we say the following?
- “Someone loves everyone”
- “There is someone who doesn’t love anyone”
- “There is someone who is not loved by anyone”
- “Everyone loves everyone”
- “There is someone who doesn’t love herself/himself.”

Understanding Quantifiers

Which formulas are true/false? If false, give a counterexample
- \( \forall x. \exists y. (\text{sameShape}(x, y) \land \text{sameColor}(x, y)) \)
- \( \forall x. \exists y. (\text{sameColor}(x, y) \land \neg \text{differentColor}(x, y)) \)
- \( \forall x. (\text{triangle}(x) \rightarrow (\exists y. (\text{circle}(y) \land \text{sameColor}(x, y)))) \)

Understanding Quantifiers, cont.

Which formulas are true/false? If false, give a counterexample
- \( \exists y. \forall x. (\text{triangle}(x) \land \text{square}(y)) \rightarrow \text{sameColor}(x, y) \)
- \( \exists x. \forall y. \neg \text{sameShape}(x, y) \)
- \( \forall x. (\text{circle}(x) \rightarrow (\exists y. (\neg \text{circle}(y) \land \text{sameColor}(x, y)))) \)

Translating First-Order Logic into English

Given predicates \( \text{student}(x) \), \( \text{atUT}(x) \), and \( \text{friends}(x, y) \), what do the following formulas say in English?
- \( \forall x. ((\text{atUT}(x) \land \text{student}(x)) \rightarrow (\exists y. (\text{friends}(x, y) \land \neg \text{atUT}(y)))) \)
- \( \forall x. ((\text{student}(x) \land \neg \text{atUT}(x)) \rightarrow \neg \exists y. \text{friends}(x, y)) \)
- \( \forall x. \forall y. ((\text{student}(x) \land \text{student}(y) \land \text{friends}(x, y)) \rightarrow (\text{atUT}(x) \land \text{atUT}(y))) \)
Translating English into First-Order Logic

Given predicates student(x), atUT(x), and friends(x, y), how do we express the following in first-order logic?

▶ “Every UT student has a friend”
▶ “At least one UT student has no friends”
▶ “All UT students are friends with each other”

Satisfiability, Validity in FOL

▶ The concepts of satisfiability, validity also important in FOL
▶ An FOL formula $F$ is satisfiable if there exists some interpretation such that $F$ evaluates to true
▶ Example: Prove that $\forall x. P(x) \rightarrow Q(x)$ is satisfiable.
▶ An FOL formula $F$ is valid if, for all interpretations, $F$ evaluates to true
▶ Prove that $\forall x. P(x) \rightarrow Q(x)$ is not valid.
▶ Formulas that are satisfiable, but not valid are contingent, e.g., $\forall x. P(x) \rightarrow Q(x)$

Equivalence

▶ Two formulas $F_1$ and $F_2$ are equivalent if $F_1 \leftrightarrow F_2$ is valid
▶ In PL, we could prove equivalence using truth tables, but not possible in FOL
▶ However, we can still use known equivalences to rewrite one formula as the other
▶ Example: Prove that $\neg(\forall x. (P(x) \rightarrow Q(x)))$ and $\exists x. (P(x) \land \neg Q(x))$ are equivalent.
▶ Example: Prove that $\neg \exists x. \forall y. P(x, y)$ and $\forall x. \exists y. \neg P(x, y)$ are equivalent.