CS311H: Discrete Mathematics
Introduction to First-Order Logic
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Announcements

- Homework due at the beginning of next lecture
- Please bring a hard copy of solutions to class!

Review: Proving Equivalence

- Prove that $\neg(p \rightarrow q)$ is equivalent to $p \land q$ using known equivalences.

Formalizing English Arguments in Logic

- We can use logic to prove/disprove arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let \( f \) be the proposition "Joe drives fast", and \( t \) be the proposition "Joe gets a ticket".
- How do you prove using logic that above argument is valid?

Another Example

- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Is this argument valid? Prove/disprove using logic!
- Let \( u \) = "Jill is carrying an umbrella", and \( r \) = "It is raining"
- How do we encode this argument in logic?

Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining."

- How can we prove George’s argument is invalid?
Why First-Order Logic?

- So far, we studied the simplest logic: propositional logic, but, in many cases, it’s not expressive enough.
- Consider the statement “Anyone who drives fast gets a speeding ticket”
- From this, we should be able to conclude “If Joe drives fast, he will get a speeding ticket”
- But PL does not allow inferences like that because we cannot talk about concepts like “everyone”, “someone” etc.
- First-order logic (predicate logic) is more expressive and allows such inferences.

Building Blocks of First-Order Logic

- The building blocks of propositional logic were propositions.
- In first-order logic, there are three kinds of basic building blocks: constants, variables, predicates.
- Constants: refer to specific objects (in a universe of discourse)
  - Examples: 6, Austin, CS311, ...
- Variables: range over objects (in a universe of discourse)
  - Examples: x, y, z, ...
- If universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio, ...

Building Blocks of First-Order Logic, cont.

- Predicates describe properties of objects or relationships between objects.
- Examples: ishappy, betterthan, loves, > ...
- Predicates can be applied to both constants and variables.
- Examples: ishappy(George), betterthan(x, y), loves(George, Rachel), x > 3, ...
- A predicate \( P(x) \) is true or false depending on whether property \( P \) holds for \( x \).
- Example: ishappy(George) is true if George is happy, but false otherwise.

Predicate Examples

- Suppose \( Q(x, y) \) denotes \( x = y + 3 \).
- What is the truth value of \( Q(3, 0) \)?
- What is the truth value of \( Q(1, 2) \)?
- What is the truth value of \( Q(x, 2) \)?

Formulas in First Order Logic

- Formulas in first-order logic are formed using predicates and logical connectives.
- Example: even(x) is also a formula.
- Example: even(x) \( \lor \) odd(x) is also a formula.
- Example: \((\text{odd}(x) \rightarrow \neg \text{even}(x)) \land \text{even}(x)\)
More Universal Quantifiers

- Universal quantification of $P(x)$, $\forall x. P(x)$, is the statement "$P(x)$ holds for all objects $x$ in the universe of discourse." 
- $\forall x. P(x)$ is true if predicate $P$ is true for every object in the universe of discourse, and false otherwise
- Consider domain $D = \{0, \star\}$, $P(0) = true, P(\star) = false$
- What is truth value of $P(x) \land Q(x)$ under $I$?
- What is truth value of $P(y, x) \rightarrow Q(y)$ under $I$?
- What is truth value of $P(x) \rightarrow Q(x)$ under $I$?
- What is truth value of $P(x, y) \land Q(x)$ under $I$?

More Universal Quantifier Examples

- Consider the domain $D$ of real numbers and predicate $P(x)$ with interpretation $x^2 \geq x$
- What is the truth value of $\forall x. P(x)$?
- What is a counterexample?
- What if the domain is integers?
- Observe: Truth value of a formula depends on a universe of discourse!

Quantifiers

- Real power of first-order logic over propositional logic: quantifiers
- Quantifiers allow us to talk about all objects or the existence of some object
- There are two quantifiers in first-order logic:
  1. Universal quantifier ($\forall$): refers to all objects
  2. Existential quantifier ($\exists$): refers to some object

Existential Quantifier Examples

- Consider the domain of reals and predicate $P(x)$ with interpretation $x < 0$.
- What is the truth value of $\exists x. P(x)$?
- What if domain is positive integers?
- Let $Q(y)$ be the statement $y > y^2$
- What’s truth value of $\exists y. Q(y)$ if domain is reals?
- What about if domain is integers?
Quantifiers Summary

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x. P(x)$</td>
<td>$P(x)$ is true for every $x$</td>
<td>$P(x)$ is false for some $x$</td>
</tr>
<tr>
<td>$\exists x. P(x)$</td>
<td>$P(x)$ is true for some $x$</td>
<td>$P(x)$ is false for every $x$</td>
</tr>
</tbody>
</table>

- Consider finite universe of discourse with objects $a_1, \ldots, a_n$.
- $\forall x. P(x)$ is true iff $P(a_1) \land P(a_2) \land \ldots \land P(a_n)$ is true
- $\exists x. P(x)$ is true iff $P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)$ is true

Quantified Formulas

- So far, only discussed how to quantify individual predicates, but can also quantify entire formulas.
- $\exists x. (\text{even}(x) \land \text{gt}(x, 100))$ is a valid formula in FOL.
- What’s truth value of this formula if domain is all integers?
  - assuming $\text{even}(x)$ means “$x$ is even” and $\text{gt}(x, y)$ means $x > y$
  - What about $\forall x. (\text{even}(x) \land \text{gt}(x, 100))$?

DeMorgan’s Laws for Quantifiers

- Learned about DeMorgan’s laws for propositional logic:
  - $\neg (p \land q) \equiv \neg p \lor \neg q$
  - $\neg (p \lor q) \equiv \neg p \land \neg q$
- DeMorgan’s laws extend to first-order logic, e.g., $\neg (\text{even}(x) \lor \text{div}4(x)) \equiv (\neg \text{even}(x) \land \neg \text{div}4(x))$
- Two new DeMorgan’s laws for quantifiers:
  - $\neg \forall x. P(x) \equiv \exists x. \neg P(x)$
  - $\neg \exists x. P(x) \equiv \forall x. \neg P(x)$
- When you push negation in, $\forall$ flips to $\exists$ and vice versa.

Using DeMorgan’s Laws

- Expressed “No one in CS311 is a freshman” as $\neg \exists x. (\text{inCS311}(x) \land \text{freshman}(x))$.
- Let’s apply DeMorgan’s law to this formula:
  - Using the fact that $p \rightarrow q$ is equivalent to $\neg p \lor q$, we can write this formula as:
  - Therefore, these two formulas are equivalent!
Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers.
- For example, can’t express “Everybody loves someone” using a single quantifier.
- Suppose predicate $\text{loves}(x, y)$ means “Person $x$ loves person $y$.”
- What does $\exists x. \forall y. \text{loves}(x, y)$ mean?
- What does $\forall y. \exists x. \text{loves}(x, y)$ mean?
- Observe: Order of quantifiers is very important!

Summary of Nested Quantifiers

<table>
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<tr>
<th>Statement</th>
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</tr>
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<tr>
<td>$\forall x. \forall y. \text{P}(x, y)$</td>
<td>$\text{P}(x, y)$ is true for every pair $x, y$.</td>
</tr>
<tr>
<td>$\forall y. \forall x. \text{P}(x, y)$</td>
<td>For every $x$, there is a $y$ for which $\text{P}(x, y)$ is true.</td>
</tr>
<tr>
<td>$\exists x. \forall y. \text{P}(x, y)$</td>
<td>There is an $x$ for which $\text{P}(x, y)$ is true for every $y$.</td>
</tr>
<tr>
<td>$\exists y. \forall x. \text{P}(x, y)$</td>
<td>There is a pair $x, y$ for which $\text{P}(x, y)$ is true.</td>
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Observe: Order of quantifiers is only important if quantifiers of different kinds!

More Nested Quantifier Examples

Using the $\text{loves}(x, y)$ predicate, how can we say the following?
- “Someone loves everyone”
- “There is someone who doesn’t love anyone”
- “There is someone who is not loved by anyone”
- “Everyone loves everyone”
- “There is someone who doesn’t love herself/himself.”

Understanding Quantifiers

Which formulas are true/false? If false, give a counterexample.
- $\forall x. \exists y. (\text{sameShape}(x, y) \land \text{differentShape}(x, y))$
- $\forall x. \exists y. (\text{sameColor}(x, y) \land \text{differentColor}(x, y))$
- $\forall x. (\text{triangle}(x) \rightarrow (\exists y. (\text{circle}(y) \land \text{sameColor}(x, y))))$

Understanding Quantifiers, cont.

Which formulas are true/false? If false, give a counterexample.
- $\forall x. \forall y. ((\text{triangle}(x) \land \text{square}(y)) \rightarrow \text{sameColor}(x, y))$
- $\exists x. \forall y. \neg \text{sameShape}(x, y)$
- $\forall x. (\text{circle}(x) \rightarrow (\exists y. (\neg \text{circle}(y) \land \text{sameColor}(x, y))))$

Translating First-Order Logic into English

Given predicates $\text{student}(x)$, $\text{atUT}(x)$, and $\text{friends}(x, y)$, what do the following formulas say in English?
- $\forall x. ((\text{atUT}(x) \land \text{student}(x)) \rightarrow (\exists y. (\text{friends}(x, y) \land \neg \text{atUT}(y))))$
- $\forall x. ((\text{student}(x) \land \neg \text{atUT}(x)) \rightarrow \neg \exists y. \text{friends}(x, y))$
- $\forall y. ((\text{student}(x) \land \text{atUT}(y)) \land \text{student}(x)) \rightarrow (\text{atUT}(y) \land \text{atUT}(x)))$
Translating English into First-Order Logic

Given predicates student(x), atUT(x), and friends(x, y), how do we express the following in first-order logic?

- "Every UT student has a friend"
- "At least one UT student has no friends"
- "All UT students are friends with each other"

Satisfiability, Validity in FOL

- The concepts of satisfiability, validity also important in FOL
- An FOL formula F is satisfiable if there exists some domain and some interpretation such that F evaluates to true
  
  **Example:** Prove that ∀x. P(x) → Q(x) is satisfiable.
- An FOL formula F is valid if, for all domains and all interpretations, F evaluates to true
  
  **Example:** Prove that ∀x. P(x) → Q(x) is not valid.
- Formulas that are satisfiable, but not valid are contingent, e.g., ∀x. P(x) → Q(x)

Equivalence

- Two formulas F1 and F2 are equivalent if F1 ↔ F2 is valid
  
  **Example:** Prove that ¬(∀x. P(x) → Q(x)) and ∃x. (P(x) ∧ ¬Q(x)) are equivalent.
- **Example:** Prove that ¬∃x. y. P(x, y) and ∀x. ∃y. ¬P(x, y) are equivalent.