Why First-Order Logic?

- So far, we studied the simplest logic: **propositional logic**
- But for some applications, propositional logic is not expressive enough
- First-order logic is more expressive: allows representing more complex facts and making more sophisticated inferences

A Motivating Example

- For instance, consider the statement "Anyone who drives fast gets a speeding ticket"
- From this, we should be able to conclude "If Joe drives fast, he will get a speeding ticket"
- Similarly, we should be able to conclude "If Rachel drives fast, she will get a speeding ticket" and so on.
- But Propositional Logic does not allow inferences like that because we cannot talk about concepts like "everyone", "someone" etc.
- **First-order logic** (predicate logic) allows making such kinds of inferences

Building Blocks of First-Order Logic

- The building blocks of propositional logic were **propositions**
- In first-order logic, there are three kinds of basic building blocks: constants, variables, predicates
- **Constants**: refer to specific objects (in a universe of discourse)
  - Examples: George, 6, Austin, CS311, . . .
- **Variables**: range over objects (in a universe of discourse)
  - Examples: x,y,z, . . .
- If universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio, . . .

Building Blocks of First-Order Logic, cont.

- **Predicates** describe properties of objects or relationships between objects
- **Examples**: ishappy, betterthan, loves, > . . .
- Predicates can be applied to both constants and variables
- **Examples**: ishappy(George), betterthan(x,y), loves(George, Rachel), x > 3, . . .
- A predicate $P(x)$ is true or false depending on whether property $P$ holds for $x$
- **Example**: ishappy(George) is true if George is happy, but false otherwise

Predicate Examples

- Consider predicate **even** which represents if a number is even
- What is truth value of **even(2)**?
- What is truth value of **even(5)**?
- What is truth value of **even(x)**?
- **Another example**: Suppose $Q(x, y)$ denotes $x = y + 3$
- What is the truth value of $Q(3, 0)$?
- What is the truth value of $Q(1, 2)$?
Formulas in First Order Logic

- Formulas in first-order logic are formed using predicates and logical connectives.
- Example: even(2) is a formula
- Example: even(x) is also a formula
- Example: even(x) ∨ odd(x) is also a formula
- Example: (odd(x) → ¬ even(x)) ∧ even(x)

Semantics of First-Order Logic

- In propositional logic, the truth value of a formula depends on a truth assignment to variables.
- In FOL, the truth value of a formula depends on an interpretation of predicate symbols and variables over some domain \( D \).
- Consider a FOL formula \( \neg P(x) \).
- A possible interpretation:
  \[ D = \{ \star, \circ \}, P(\star) = \text{true}, P(\circ) = \text{false}, x = \star \]
- Under this interpretation, what’s the truth value of \( \neg P(x) \)?
- What about if \( x = \circ \)?

More Examples

- Consider interpretation \( I \) over domain \( D = \{ 1, 2 \} \):
  - \( P(1, 1) = P(1, 2) = \text{true}, P(2, 1) = P(2, 2) = \text{false} \)
  - \( Q(1) = \text{false}, Q(2) = \text{true} \)
  - \( x = 1, y = 2 \)
- What is truth value of \( P(x, y) \land Q(y) \) under \( I \)?
- What is truth value of \( P(y, x) \rightarrow Q(y) \) under \( I \)?
- What is truth value of \( P(x, y) \rightarrow Q(x) \) under \( I \)?

Quantifiers

- Real power of first-order logic over propositional logic: quantifiers
- Quantifiers allow us to talk about all objects or the existence of some object
- There are two quantifiers in first-order logic:
  1. Universal quantifier (\( \forall \)): refers to all objects
  2. Existential quantifier (\( \exists \)): refers to some object

Universal Quantifiers

- Universal quantification of \( P(x) \), \( \forall x. P(x) \), is the statement “\( P(x) \) holds for all objects \( x \) in the universe of discourse.”
- \( \forall x. P(x) \) is true if predicate \( P \) is true for every object in the universe of discourse, and false otherwise
- Consider domain \( D = \{ \star, \circ \} \), \( P(\circ) = \text{true}, P(\star) = \text{false} \)
- What is truth value of \( \forall x. P(x) \)?
- Object \( o \) for which \( P(o) \) is false is a counterexample of \( \forall x. P(x) \)
- What is a counterexample for \( \forall x. P(x) \) in previous example?

More Universal Quantifier Examples

- Consider the domain \( D \) of real numbers and predicate \( P(x) \) with interpretation \( x^2 \geq x \).
- What is the truth value of \( \forall x. P(x) \)?
- What is a counterexample?
- What if the domain is integers?
- Observe: Truth value of a formula depends on a universe of discourse!
Existential Quantifiers

- **Existential quantification** of $P(x)$, written $\exists x. P(x)$, is "There exists an element $x$ in the domain such that $P(x)$ is true.
- $\exists x. P(x)$ is true if there is at least one element in the domain such that $P(x)$ is true.
- In first-order logic, domain is required to be **non-empty**.
- Consider domain $D = \{ o, * \}$, $P(o) = \text{true}$, $P(*) = \text{false}$.
- What is truth value of $\exists x. P(x)$?

Existential Quantifier Examples

- Consider the domain of reals and predicate $P(x)$ with interpretation $x < 0$.
- What is the truth value of $\exists x. P(x)$?
- What if domain is positive integers?
- Let $Q(y)$ be the statement $y > y^2$.
- What's truth value of $\exists y. Q(y)$ if domain is reals?
- What about if domain is integers?

More Examples of Quantified Formulas

- Consider the domain of integers and the predicates $\text{even}(x)$ and $\text{div4}(x)$ which represents if $x$ is divisible by 4.
- What is the truth value of the following quantified formulas?
  - $\forall x. (\text{div4}(x) \rightarrow \text{even}(x))$
  - $\forall x. (\text{even}(x) \rightarrow \text{div4}(x))$
  - $\exists x. (\neg \text{div4}(x) \land \text{even}(x))$
  - $\exists x. (\neg \text{div4}(x) \rightarrow \text{even}(x))$
  - $\forall x. (\neg \text{div4}(x) \rightarrow \text{even}(x))$

Quantifiers Summary

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
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- Consider finite universe of discourse with objects $o_1, \ldots, o_n$.
- $\forall x. P(x)$ is true iff $P(o_1) \land P(o_2) \land \ldots \land P(o_n)$ is true.
- $\exists x. P(x)$ is true iff $P(o_1) \lor P(o_2) \lor \ldots \lor P(o_n)$ is true.

Quantified Formulas

- So far, only discussed how to quantify individual predicates.
- But we can also quantify entire formulas containing multiple predicates and logical connectives.
- $\exists x. (\text{even}(x) \land \text{gt}(x, 100))$ is a valid formula in FOL.
- What’s truth value of this formula if domain is all integers?
  - assuming $\text{even}(x)$ means "$x$ is even" and $\text{gt}(x, y)$ means $x > y$.
  - What about $\forall x. (\text{even}(x) \land \text{gt}(x, 100))$?

Translating English Into Quantified Formulas

Assuming $\text{freshman}(x)$ means "$x$ is a freshman" and $\text{inCS311}(x)$ "$x$ is taking CS311", express the following in FOL.

- Someone in CS311 is a freshman.
- No one in CS311 is a freshman.
- Everyone taking CS311 are freshmen.
- Every freshman is taking CS311.
DeMorgan’s Laws for Quantifiers

- Learned about DeMorgan’s laws for propositional logic:

  \[ \neg(p \land q) \equiv \neg p \lor \neg q \]
  \[ \neg(p \lor q) \equiv \neg p \land \neg q \]
  
- DeMorgan’s laws extend to first-order logic, e.g.,

  \[ \neg(even(x) \lor div4(x)) \equiv (\neg even(x) \land \neg div4(x)) \]

- Two new DeMorgan’s laws for quantifiers:

  \[ \neg \forall x. P(x) \equiv \exists x. \neg P(x) \]
  \[ \neg \exists x. P(x) \equiv \forall x. \neg P(x) \]

- When you push negation in, \( \forall \) flips to \( \exists \) and vice versa.

Using DeMorgan’s Laws

- Expressed “No one in CS311 is a freshman” as

  \[ \neg \exists x. (\text{inCS311}(x) \land \text{freshman}(x)) \]

- Let’s apply DeMorgan’s law to this formula:

  Using the fact that \( p \to q \) is equivalent to \( \neg p \lor q \), we can write this formula as:

  \[ \neg \exists x. \neg \text{inCS311}(x) \lor \neg \text{freshman}(x) \]

  Therefore, these two formulas are equivalent!

Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers

- For example, can’t express “Everybody loves someone” using a single quantifier

- Suppose predicate \( \text{loves}(x, y) \) means “Person \( x \) loves person \( y \)”

- What does \( \forall x. \exists y. \text{loves}(x, y) \) mean?

- What does \( \exists y. \forall x. \text{loves}(x, y) \) mean?

- Observe: Order of quantifiers is very important!

More Nested Quantifier Examples

Using the \( \text{loves}(x, y) \) predicate, how can we say the following?

- “Someone loves everyone”

- “There is someone who doesn’t love anyone”

- “There is someone who is not loved by anyone”

- “Everyone loves everyone”

- “There is someone who doesn’t love herself/himself.”

Summary of Nested Quantifiers

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Observe: Order of quantifiers is only important if quantifiers of different kinds!

Understanding Quantifiers

Which formulas are true/false? If false, give a counterexample

- \( \forall x. \exists y. (\text{sameShape}(x, y) \land \text{differentColor}(x, y)) \)

- \( \forall x. \exists y. (\text{sameColor}(x, y) \land \text{differentShape}(x, y)) \)

- \( \forall x. (\text{triangle}(x) \to (\exists y. (\text{circle}(y) \land \text{sameColor}(x, y)))) \)
Translating First-Order Logic into English

Given predicates \( \text{student}(x) \), \( \text{atUT}(x) \), and \( \text{friends}(x, y) \), what do the following formulas say in English?

- \( \forall x. ((\text{atUT}(x) \land \text{student}(x)) \rightarrow (\exists y. (\text{friends}(x, y) \land \neg \text{atUT}(y)))) \)
- \( \forall x. ((\text{student}(x) \land \neg \text{atUT}(x)) \rightarrow \neg \exists y. \text{friends}(x, y)) \)
- \( \forall x. \forall y. ((\text{student}(x) \land \text{student}(y) \land \text{friends}(x, y)) \rightarrow (\text{atUT}(x) \land \text{atUT}(y))) \)

Satisfiability, Validity in FOL

- The concepts of satisfiability, validity also important in FOL
- An FOL formula \( F \) is satisfiable if there exists some domain and some interpretation such that \( F \) evaluates to true
- Example: Prove that \( \forall x. P(x) \rightarrow Q(x) \) is satisfiable.
- An FOL formula \( F \) is valid if, for all domains and all interpretations, \( F \) evaluates to true
- Prove that \( \forall x. P(x) \rightarrow Q(x) \) is not valid.
- Formulas that are satisfiable, but not valid are contingent, e.g., \( \forall x. P(x) \rightarrow Q(x) \)

Understanding Quantifiers, cont.

Which formulas are true/false? If false, give a counterexample

- \( \forall x. \forall y. ((\text{triangle}(x) \land \text{square}(y)) \rightarrow \text{sameColor}(x, y)) \)
- \( \exists x. \forall y. \neg \text{sameShape}(x, y) \)
- \( \forall x. (\text{circle}(x) \rightarrow (\exists y. (\neg \text{circle}(y) \land \text{sameColor}(x, y)))) \)

Translating English into First-Order Logic

Given predicates \( \text{student}(x) \), \( \text{atUT}(x) \), and \( \text{friends}(x, y) \), how do we express the following in first-order logic?

- “Every UT student has a friend”
- “At least one UT student has no friends”
- “All UT students are friends with each other”

Equivalence

- Two formulas \( F_1 \) and \( F_2 \) are equivalent if \( F_1 \leftrightarrow F_2 \) is valid
- In PL, we could prove equivalence using truth tables, but not possible in FOL
- However, we can still use known equivalences to rewrite one formula as the other
- Example: Prove that \( \neg (\forall x. (P(x) \rightarrow Q(x))) \) and \( \exists x. (P(x) \land \neg Q(x)) \) are equivalent.
- Example: Prove that \( \neg \exists x. \forall y. P(x, y) \) and \( \forall x. \exists y. \neg P(x, y) \) are equivalent.