Why First-Order Logic?

- So far, we studied the simplest logic: **propositional logic**
- But for some applications, propositional logic is not expressive enough
- First-order logic is more expressive: allows representing more complex facts and making more sophisticated inferences

A Motivating Example

- For instance, consider the statement “Anyone who drives fast gets a speeding ticket”
- From this, we should be able to conclude “If Joe drives fast, he will get a speeding ticket”
- Similarly, we should be able to conclude “If Rachel drives fast, she will get a speeding ticket” and so on.
- But PL does not allow inferences like that because we cannot talk about concepts like “everyone”, “someone” etc.
- **First-order logic** (predicate logic) allows making such kinds of inferences

Building Blocks of First-Order Logic

- The building blocks of propositional logic were **propositions**
- In first-order logic, there are three kinds of basic building blocks: constants, variables, predicates
- **Constants**: refer to specific objects (in a universe of discourse)
  - Examples: George, 6, Austin, CS311, . . .
- **Variables**: range over objects (in a universe of discourse)
  - Examples: x,y,z, . . .
- If universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio, . . .

Building Blocks of First-Order Logic, cont.

- **Predicates** describe properties of objects or relationships between objects
- **Examples**: ishappy, betterthan, loves, > . . .
- Predicates can be applied to both constants and variables
- **Examples**: ishappy(George), betterthan(x,y), loves(George, Rachel), x > 3, . . .
- A predicate $P(c)$ is true or false depending on whether property $P$ holds for $c$
- **Example**: ishappy(George) is true if George is happy, but false otherwise

Predicate Examples

- Consider predicate **even** which represents if a number is even
- What is truth value of **even(2)**?
- What is truth value of **even(5)**?
- What is truth value of **even(x)**?
- Another example: Suppose $Q(x, y)$ denotes $x = y + 3$
- What is the truth value of **$Q(3, 0)$**?
- What is the truth value of **$Q(1, 2)$**?
Formulas in First Order Logic

- Formulas in first-order logic are formed using predicates and logical connectives.
- Example: even(2) is a formula
- Example: even(x) is also a formula
- Example: even(x) ∨ odd(x) is also a formula
- Example: (odd(x) → ¬ even(x)) ∧ even(x)

Semantics of First-Order Logic

- In propositional logic, the truth value of a formula depends on a truth assignment to variables.
- In FOL, truth value of a formula depends on interpretation of predicate symbols and variables over some domain D.
- Consider a FOL formula ¬P(x).
- A possible interpretation:
  \[ D = \{*, \circ\}, P(*) = true, P(\circ) = false, x = * \]
- Under this interpretation, what’s truth value of ¬P(x)?
- What about if x = o?

More Universal Quantifier Examples

- Consider the domain D of real numbers and predicate P(x) with interpretation \[ x^2 \geq x \]
- What is the truth value of \( \forall x.P(x) \)?
- What is a counterexample?
- What if the domain is integers?
- Observe: Truth value of a formula depends on a universe of discourse!

More Examples

- Consider interpretation I over domain D = \{1, 2\}
  - P(1, 1) = P(1, 2) = true, P(2, 1) = P(2, 2) = false
  - Q(1) = false, Q(2) = true
  - x = 1, y = 2
- What is truth value of \( P(x, y) \land Q(y) \) under I?
- What is truth value of \( P(y, x) \rightarrow Q(y) \) under I?
- What is truth value of \( P(x, y) \rightarrow Q(x) \) under I?

Quantifiers

- Real power of first-order logic over propositional logic: quantifiers
- Quantifiers allow us to talk about all objects or the existence of some object.
- There are two quantifiers in first-order logic:
  1. Universal quantifier (\( \forall \)): refers to all objects
  2. Existential quantifier (\( \exists \)): refers to some object

Universal Quantifiers

- Universal quantification of \( P(x) \), \( \forall x.P(x) \), is the statement "P(x) holds for all objects x in the universe of discourse."
- \( \forall x.P(x) \) is true if predicate P is true for every object in the universe of discourse, and false otherwise.
- Consider domain D = \{*, \circ\}, P(\circ) = true, P(*) = false
- What is truth value of \( \forall x.P(x) \)?
- Object o for which \( P(o) \) is false is a counterexample of \( \forall x.P(x) \)
- What is a counterexample for \( \forall x.P(x) \) in previous example?
### Existential Quantifiers

- **Existential quantification** of $P(x)$, written $\exists x. P(x)$, is "There exists an element $x$ in the domain such that $P(x)$".
- $\exists x. P(x)$ is true if there is at least one element in the domain such that $P(x)$ is true.
- In first-order logic, domain is required to be non-empty.
- Consider domain $D = \{0, \star\}$, $P(0) = \text{true}$, $P(\star) = \text{false}$.
- What is truth value of $\exists x. P(x)$?

### Existential Quantifier Examples

- Consider the domain of integers and the predicates $\text{even}(x)$ and $\text{div}4(x)$ which represents if $x$ is divisible by 4.
- $\exists x. (\neg\text{div}4(x) \land \text{even}(x))$ is true if there is at least one element in the domain such that $\neg\text{div}4(x)$ and $\text{even}(x)$ are both true.

### Quantifiers Summary

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x. P(x)$</td>
<td>$P(x)$ is true for every $x$</td>
<td>$P(x)$ is false for some $x$</td>
</tr>
<tr>
<td>$\exists x. P(x)$</td>
<td>$P(x)$ is true for some $x$</td>
<td>$P(x)$ is false for every $x$</td>
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</table>

- Consider finite universe of discourse with objects $o_1, \ldots, o_n$.
- $\forall x. P(x)$ is true iff $P(o_1) \land P(o_2) \ldots \land P(o_n)$ is true.
- $\exists x. P(x)$ is true iff $P(o_1) \lor P(o_2) \ldots \lor P(o_n)$ is true.

### More Examples of Quantified Formulas

- Consider the domain of integers and the predicates $\text{even}(x)$ and $\text{div}4(x)$ which represents if $x$ is divisible by 4.
- What is the truth value of the following quantified formulas?
  - $\forall x. (\neg\text{div}4(x) \rightarrow \text{even}(x))$
  - $\forall x. (\text{even}(x) \rightarrow \text{div}4(x))$
  - $\exists x. (\neg\text{div}4(x) \land \text{even}(x))$
  - $\exists x. (\neg\text{div}4(x) \rightarrow \text{even}(x))$
  - $\forall x. (\neg\text{div}4(x) \rightarrow \text{even}(x))$

### Quantified Formulas

- So far, only discussed how to quantify individual predicates.
- But we can also quantify entire formulas containing multiple predicates and logical connectives.
- $\exists x. (\text{even}(x) \land \text{gt}(x, 100))$ is a valid formula in FOL.
- What’s truth value of this formula if domain is all integers?
  - assuming $\text{even}(x)$ means "$x$ is even" and $\text{gt}(x, y)$ means $x > y$.
  - What about $\forall x. (\text{even}(x) \land \text{gt}(x, 100))$?

### Translating English Into Quantified Formulas

Assuming $\text{freshman}(x)$ means "$x$ is a freshman" and $\text{inCS311}(x)$ "$x$ is taking CS311", express the following in FOL.

- Someone in CS311 is a freshman.
- No one in CS311 is a freshman.
- Everyone taking CS311 are freshmen.
- Every freshman is taking CS311.
De Morgan’s Laws for Quantifiers

- Learned about De Morgan’s laws for propositional logic:
  \[
  \neg(p \land q) \equiv \neg p \lor \neg q \\
  \neg(p \lor q) \equiv \neg p \land \neg q 
  \]
- De Morgan’s laws extend to first-order logic, e.g.,
  \[\neg(even(x) \lor div_4(x)) \equiv (\neg even(x) \land \neg div_4(x))\]
- Two new De Morgan’s laws for quantifiers:
  \[\neg \forall x. P(x) \equiv \exists x. \neg P(x) \]
  \[\neg \exists x. P(x) \equiv \forall x. \neg P(x)\]
- When you push negation in, \(\forall\) flips to \(\exists\) and vice versa

Using De Morgan’s Laws

- Expressed “No one in CS311 is a freshman” as
  \[\neg \exists x. (\text{inCS311}(x) \land \text{freshman}(x))\]
- Let’s apply De Morgan’s law to this formula:
  \[\neg (\text{inCS311}(x) \lor \text{freshman}(x))\]
- Using the fact that \(p \to q\) is equivalent to \(\neg p \lor q\), we can write this formula as:
  \[\neg p \lor \neg q\]
- Therefore, these two formulas are equivalent!

Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers
- For example, can’t express “Everybody loves someone” using a single quantifier
- Suppose predicate \(\text{loves}(x, y)\) means “Person \(x\) loves person \(y\)”
- What does \(\forall x. \exists y. \text{loves}(x, y)\) mean?
- What does \(\exists y. \forall x. \text{loves}(x, y)\) mean?
- Observe: Order of quantifiers is very important!

More Nested Quantifier Examples

Using the \(\text{loves}(x, y)\) predicate, how can we say the following?

- “Someone loves everyone”
- “There is someone who doesn’t love anyone”
- “There is someone who is not loved by anyone”
- “Everyone loves everyone”
- “There is someone who doesn’t love herself/himself.”

Summary of Nested Quantifiers

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<tr>
<td>\forall x. \exists y. P(x, y)</td>
<td>(P(x, y)) is true for every pair (x, y)</td>
</tr>
<tr>
<td>\exists y. \forall x. P(x, y)</td>
<td>For every (x), there is a (y) for which (P(x, y)) is true</td>
</tr>
<tr>
<td>\forall x. \exists y. P(x, y)</td>
<td>There is an (x) for which (P(x, y)) is true for every (y)</td>
</tr>
<tr>
<td>\exists x. \exists y. P(x, y)</td>
<td>There is a pair (x, y) for which (P(x, y)) is true</td>
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Observe: Order of quantifiers is only important if quantifiers of different kinds!

Understanding Quantifiers

Which formulas are true/false? If false, give a counterexample

- \(\forall x. \exists y. (\text{sameShape}(x, y) \land \text{differentColor}(x, y))\)
- \(\forall x. \exists y. (\text{sameColor}(x, y) \land \text{differentShape}(x, y))\)
- \(\forall x. (\text{triangle}(x) \to (\exists y. (\text{circle}(y) \land \text{sameColor}(x, y))))\)
Understanding Quantifiers, cont.

Which formulas are true/false? If false, give a counterexample

- \( \forall x. \forall y. ((\triangle(x) \land \square(y)) \rightarrow \text{sameColor}(x, y)) \)
- \( \exists x. \forall y. \neg \text{shape}(x, y) \)
- \( \forall x. (\text{circle}(x) \rightarrow (\exists y. (\neg \text{circle}(y) \land \text{sameColor}(x, y)))) \)

Translating English into First-Order Logic

Given predicates \( \text{student}(x) \), \( \text{atUT}(x) \), and \( \text{friends}(x, y) \), how do we express the following in first-order logic?

- “Every UT student has a friend”
- “At least one UT student has no friends”
- “All UT students are friends with each other”

Satisfiability, Validity in FOL

- The concepts of satisfiability, validity also important in FOL
- An FOL formula \( F \) is satisfiable if there exists some domain and some interpretation such that \( F \) evaluates to true
- Example: Prove that \( \forall x. P(x) \rightarrow Q(x) \) is satisfiable.
- An FOL formula \( F \) is valid if, for all domains and all interpretations, \( F \) evaluates to true
- Prove that \( \forall x. P(x) \rightarrow Q(x) \) is not valid.
- Formulas that are satisfiable, but not valid are contingent, e.g., \( \forall x. P(x) \rightarrow Q(x) \)

Equivalence

- Two formulas \( F_1 \) and \( F_2 \) are equivalent if \( F_1 \leftrightarrow F_2 \) is valid
- In PL, we could prove equivalence using truth tables, but not possible in FOL
- However, we can still use known equivalences to rewrite one formula as the other
- Example: Prove that \( \neg (\forall x. (P(x) \rightarrow Q(x))) \) and \( \exists x. (P(x) \land \neg Q(x)) \) are equivalent.
- Example: Prove that \( \neg \exists x. \forall y. P(x, y) \) and \( \forall x. \exists y. \neg P(x, y) \) are equivalent.