Hoare Logic, Part II

Işıl Dillig

Proof Rule for While and Loop Invariants

- Last proof rule of Hoare logic is that for `while` loops.
- But to understand proof rule for while, we first need concept of a loop invariant
- A loop invariant I has following properties:
  1. I holds initially before the loop
  2. I holds after each iteration of the loop

Examples

- Consider the following code

  $\mathtt{i} := 0;\ \mathtt{j} := 0;\ \mathtt{n} := 10;\ \mathtt{while}\ \mathtt{i} < \mathtt{n}\ \mathtt{do}\ \mathtt{i} := \mathtt{i} + 1;\ \mathtt{j} := \mathtt{i} + \mathtt{j}$

- Which of the following are loop invariants?
  - $i \leq n$ yes
  - $i < n$ no
  - $j \geq 0$ yes
- Suppose I is a loop invariant. Does I also hold after loop terminates?
- Yes because, by definition, I holds after every loop iteration, including after the last one

Proof Rule for While

- Consider the statement $\mathtt{while}\ C\ \mathtt{do}\ S$
- Suppose I is a loop invariant for this loop. What is guaranteed to hold after loop terminates? $I \land \neg C$
- Putting all this together, proof rule for while is:

$$\frac{\vdash \{P \land C\} S \{P\}}{\vdash \{P\} \mathtt{while}\ C\ \mathtt{do}\ S \{P \land \neg C\}}$$

- This rule simply says "If P is a loop invariant, then $P \land \neg C$ must hold after loop terminates"
- Based on this rule, why is P a loop invariant?

Example

- Consider the statement $S = \mathtt{while}\ x < n\ \mathtt{do}\ x = x + 1$
- Let's prove validity of $\{x \leq n\} S \{x \geq n\}$
- What is appropriate loop invariant? $x \leq n$
- First, let's prove $x \leq n$ is loop invariant. What do we need to show? $\{x \leq n \land x < n\} x = x + 1 \{x \leq n\}$
- What proof rules do we need to use to show this? assignment, precondition strengthening

$$\frac{\vdash \{x \leq n[x+1/x]\} x = x + 1 \{x \leq n\} \vdash \{x + 1 \leq n\} x = x + 1 \{x}{\vdash \{x \leq n \land x < n\} x = x + 1 \{x \leq}$$

Example, cont

- Ok, we've shown $x \leq n$ is loop invariant, now let's instantiate proof rule for while with this loop invariant:

$$\frac{\vdash \{x \leq n \land x < n\} S' \{x \leq n\}}{\vdash \{x \leq n\} \mathtt{while}\ x < n\ \mathtt{do}\ S' \{x \leq n \land \neg(x < n)\}}$$

- Recall: We wanted to prove the Hoare triple $\{x \leq n\} S \{x \geq n\}$
- In addition to proof rule for while, what other rule do we need? postcondition weakening

Example, cont.

The full proof:

$$\dfrac{\dfrac{\vdash \{x+1 \leq n\} x = x+1 \{x \leq n\} \quad x \leq n \land x < n \Rightarrow x+1 < n}{\vdash \{x \leq n \land x < n\} x = x+1 \{x \leq n\}}}{\vdash \{x \leq n\} S \{x \leq n \land \neg(x < n)\}} \quad x \leq n \land \neg(x<n) \Rightarrow x \geq n$$
$$\{x \leq n\} S \{x \geq n\}$$

Invariant vs. Inductive Invariant

- Suppose I is a loop invariant for `while C do S`.
- Does it always satisfy $\{I \land C\} S \{I\}$?
- Counterexample: Consider $I = j \geq 1$ and the code:

  i := 1; j := 1; while i < n do {j := j + i; i := i + 1}
- But strengthened invariant $j \geq 1 \land i \geq 1$ does satisfy it
- Such invariants are called inductive invariants, and they are the only invariants that we can prove
- Key challenge in verification is finding inductive loop invariants

Exercise

Find inductive loop invariant to prove the following Hoare triple:

$\{i = 0 \land j = 0 \land n = 5\}$
while i < n do i := i + 1; j := j + i
$\{j = 15\}$

- Inductive loop invariant I:

  $$2j = i(i+1) \land i \leq n \land n = 5$$
- Weakest precondition P w.r.t loop body:

  $$2j = i(i+1) \land i+1 \leq n \land n = 5$$
- Since $I \land C \Rightarrow P$, I is inductive.

Summary of Proof Rules

1. $\vdash \{Q[E/x]\}\ x = E\ \{Q\}$ (Assignment)
2. $\dfrac{\vdash \{P'\}S\{Q\} \quad P \Rightarrow P'}{\vdash \{P\}S\{Q\}}$ (Strengthen P)
3. $\dfrac{\vdash \{P\}S\{Q'\} \quad Q' \Rightarrow Q}{\vdash \{P\}S\{Q\}}$ (Weaken Q)
4. $\dfrac{\vdash \{P\}C_1\{Q\} \quad \vdash \{Q\}C_2\{R\}}{\vdash \{P\}C_1; C_2\{R\}}$ (Composition)
5. $\dfrac{\vdash \{P \land C\}\ S_1\ \{Q\} \quad \vdash \{P \land \neg C\}\ S_2\ \{Q\}}{\vdash \{P\}\ \texttt{if}\ C\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{Q\}}$ (If)
6. $\dfrac{\vdash \{P \land C\}S\{P\}}{\vdash \{P\}\texttt{while}\ C\ \texttt{do}\ S\{P \land \neg C\}}$ (While)

Meta-theory: Soundness of Proof Rules

- It can be show that the proof rules for Hoare logic are sound:

  $$\text{If } \vdash \{P\}S\{Q\}, \text{ then } \models \{P\}S\{Q\}$$
- That is, if a Hoare triple $\{P\}S\{Q\}$ is provable using the proof rules, then $\{P\}S\{Q\}$ is indeed valid
- Completeness of proof rules means that if $\{P\}S\{Q\}$ is a valid Hoare triple, then it can be proven using our proof rules, i.e.,

  $$\text{If } \models \{P\}S\{Q\}, \text{ then } \vdash \{P\}S\{Q\}$$
- Unfortunately, completeness does not hold!

Meta-theory: Relative Completeness

- Recall: Rules for precondition strengthening and postcondition weakening require checking $A \Rightarrow B$
- In general, these formulas belong to Peano arithmetic
- Since PA is incomplete, there are implications that are valid but cannot be proven
- However, Hoare's proof rules still have important goodness guarantee: relative completeness
- If we have an oracle for deciding whether an implication $A \Rightarrow B$ holds, then any valid Hoare triple can be proven using our proof rules

Automating Reasoning in Hoare Logic

- Manually proving correctness is tedious, so we'd like to automate the tedious parts of program verification
- Idea: Assume an oracle gives loop invariants, but automate the rest of the reasoning
- This oracle can either be a human or a static analysis tool (e.g., abstract interpretation)

Basic Idea Behind Program Verification

- Automating Hoare logic is based on generating verification conditions (VC)
- A verification condition is a formula ϕ such that program is correct iff ϕ is valid
- Deductive verification has two components:
  1. Generate VC's from source code
  2. Use theorem prover to check validity of formulas from step 1

Generating VCs: Forwards vs. Backwards

- Two ways to generate verification conditions: forwards or backwards
- A forwards analysis starts from precondition and generates formulas to prove postcondition
- Forwards technique computes strongest postconditions (sp)
- In contrast, backwards analysis starts from postcondition and tries to prove precondition
- Backwards technique computes weakest preconditions (wp)
- We'll use the backwards method

Weakest Preconditions

- Idea: Suppose we want to verify Hoare triple $\{P\}S\{Q\}$
- We'll start with Q and going backwards, compute formula $wp(S, Q)$ called weakest precondition of Q w.r.t. to S
- $wp(S, Q)$ has the property that it is the weakest condition that guarantees Q will hold after S in any execution
- Thus, Hoare triple $\{P\}S\{Q\}$ is valid iff:

$$P \Rightarrow wp(S, Q)$$

- Why? Because if triple $\{P'\}S\{Q\}$ is valid and $P \Rightarrow P'$, then $\{P\}S\{Q\}$ is also valid

Defining Weakest Preconditions

- Weakest preconditions are defined inductively and follow Hoare's proof rules
- $wp(x := E, Q) = Q[E/x]$
- $wp(s_1; s_2, Q) = wp(s_1, wp(s_2, Q))$
- $wp(\texttt{if } C \texttt{ then } s_1 \texttt{ else } s_2, Q) = C \rightarrow wp(s_1, Q) \wedge \neg C \rightarrow wp(s_2, Q)$
- This says "If C holds, wp of then branch must hold; otherwise, wp of else branch must hold"

Example

- Consider the following code S:

$$\texttt{x} := \texttt{y} + 1;\ \texttt{if x} > 0 \texttt{ then z} := 1 \texttt{ else z} := -1$$

- What is wp(S, $z > 0$)? $y \geq 0$
- What is wp(S, $z \leq 0$)? $y < 0$
- Can we prove post-condition $z = 1$ if precondition is $y \geq -1$?
- What if precondition is $y > -1$?

Weakest Preconditions for Loops

- Unfortunately, we can't compute weakest preconditions for loops exactly...
- Idea: approximate it using $awp(S, Q)$
- $awp(S, Q)$ may be stronger than $wp(S, Q)$ but not weaker
- To verify $\{P\}S\{Q\}$, show $P \Rightarrow awp(S, Q)$
- Hope is that $awp(S, Q)$ is weak enough to be implied by P although it may not be the weakest

Approximate Weakest Preconditionsr loops, we will rely on loop invariants provided by oracle (human or static ana

- For all statements except for while loops, computation of $awp(S, Q)$ same as $wp(S, Q)$
- To compute, $awp(S, Q)$ for loops, we will rely on loop invariants provided by oracle (human or static analysis)
- Assume all loops are annotated with invariants $\texttt{while } C \texttt{ do } [I] \ S$
- Now, we'll just define $awp(\texttt{while } C \texttt{ do } [I] \ S, Q) \equiv I$
- Why is this sound? If I is an invariant, it must hold before the loop

Verification with Approximate Weakest Preconditions

- If $P \Rightarrow awp(S, Q)$, does this mean $\{P\}S\{Q\}$ is valid?
- No, two problems with $awp(\texttt{while } C \texttt{ do } \{I\} \ S, Q)$
  1. We haven't checked I is an actual loop invariant
  2. We also haven't made sure $I \land \neg C$ is sufficient to establish Q!
- For each statement S, generate verification condition $VC(S, Q)$ that encodes additional conditions to prove

Generating Verification Conditions

- Most interesting VC generation rule is for loops:

$$VC(\texttt{while } C \texttt{ do } [I] \ S, Q) = ?$$

- To ensure Q is satisfied after loop, what condition must hold? $I \land \neg C \Rightarrow Q$
- Assuming I holds initially, need to check I is loop invariant
- i.e., need to prove $\{I \land C\}S\{I\}$
- How can we prove this? check validity of $I \land C \Rightarrow awp(S, I) \ \land VC(S, I)$

Verification Condition for Loops

- To summarize, to show I is preserved in loop, need:

$$I \land C \Rightarrow awp(S, I) \ \land VC(S, I)$$

- To show I is strong enough to establish Q, need:

$$I \land \neg C \Rightarrow Q$$

- Putting this together, verification condition for a while loop $S' = \texttt{while } C \texttt{ do } \{I\} \ S$ is:

$$VC(S', Q) = (I \land C \Rightarrow awp(S, I) \land VC(S, I)) \land (I \land \neg C \Rightarrow Q)$$

Verification Condition for Other Statements

- We also need rules to generate VC's for other statements because there might be loops nested in them
- $VC(x := E, Q) = true$
- $VC(s_1; s_2, Q) = VC(s_2, Q) \land VC(s_1, awp(s_2, Q))$
- $VC(\texttt{if } C \texttt{ then } s_1 \texttt{ else } s_2, Q) = VC(s_1, Q) \land VC(s_2, Q)$

Verification of Hoare Triple

- Thus, to show validity of $\{P\}S\{Q\}$, need to do following:
  1. Compute $awp(S, Q)$
  2. Compute $VC(S, Q)$
- Theorem: $\{P\}S\{Q\}$ is valid if following formula is valid:

$$VC(S, Q) \ \wedge \ P \rightarrow awp(S, Q) \quad (*)$$

- Thus, if we can prove of validity of $(*)$, we have shown that program obeys specification

Discussion

Theorem: $\{P\}S\{Q\}$ is valid if following formula is valid:

$$VC(S, Q) \ \wedge \ P \rightarrow awp(S, Q) \quad (*)$$

- Question: If $\{P\}S\{Q\}$ is valid, is $(*)$ valid?
- No, for two reasons:
  1. Loop invariant might not be strong enough
  2. Loop invariant might be bogus
- Thus, even if program obeys specification, might not be able to prove it b/c loop invariants we use are not strong enough

Example

- Consider the following code:

```
i := 1; sum := 0;
while i ≤ n do [sum ≥ 0] {
    j := 1;
    while j ≤ i do [sum ≥ 0 ∧ j ≥ 0]
       sum := sum + j; j := j + 1
    i := i + 1
}
```

- Show the VC's generated for this program for post-condition $sum \geq 0$ – can it be verified?
- What is the post-condition we need to show for inner loop? $sum \geq 0$

Example, cont.

- Generate VC's for inner loop:

$$(1) \quad (sum \geq 0 \wedge j \geq 0 \wedge j > i) \Rightarrow sum \geq 0$$
$$(2) \quad (j \leq i \wedge sum \geq 0 \wedge j \geq 0) \Rightarrow (sum + j \geq 0 \wedge j + 1 \geq 0))$$

- Now, generate VC's for outer loop:

$$(3) \quad (i \leq n \wedge sum \geq 0) \Rightarrow (sum \geq 0 \wedge 1 \geq 0)$$
$$(4) \quad (i > n \wedge sum \geq 0) \Rightarrow sum \geq 0$$

- Finally, compute awp for outer loop: $(5) \ 0 \geq 0$
- Feed the formula $(1) \wedge (2) \wedge (3) \wedge (4) \wedge (5)$ to SMT solver
- It's valid; hence program is verified!

Example: Variant

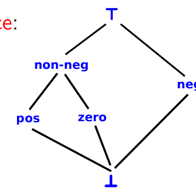
- Suppose annotated invariant for inner loop was $sum \geq 0$ instead of $sum \geq 0 \wedge j \geq 0$
- Could the program be verified then? no, because loop invariant not strong enough
- While VC generation handles many tedious aspects of the proof, user must still come up with loop invariants...

Guess-and-Check

- Fortunately, there are many automated techniques for loop invariant generation
- The simplest technique is guess-and-check
- Given template of invariants (e.g., $? = ?, \ ? \leq ?$), instantiate the holes with program variables and constants
- Then, check if it's an invariant; if not, try a different instantiation

Abstract Interpretation

- Symbolically execute the program over an abstraction until we reach a fixed point
- Example: In sign abstract domain, only track if a variable x is positive, non-negative, negative, or zero
- This defines a lattice:

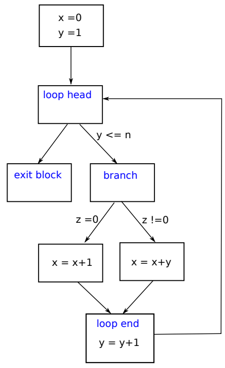


- Initialize everything to $\perp$ and then take the join of the new value with old value; repeat until you reach fixed point

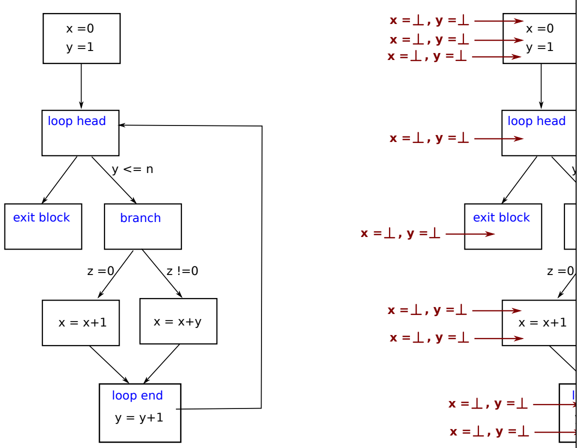
An Example

```
x = 0;
y =0;

while(y <= n)
{
  if (z == 0) {
    x = x+1;
  }
  else {
    x = x + y;
  }
  y = y+1
}
```



Fixed-Point Computation



Abstract Interpretation, cont.

- The sign abstract domain allows inferring simple invariants of the form $x \geq 0, x < 0$ etc.
- More interesting abstract domains:
  - Intervals: Tracks ranges (e.g., $x \in [0, 100]$)
  - Polyhedra: Tracks linear inequalities (e.g., $x \leq y + z$)
  - Karr's domain: Tracks linear equalities (e.g., $x = y + z$)
- In these domains, we may not reach a fixed point; apply so-called widening operation to force fixed-point

Conclusion

- Program verification automates reasoning about program correctness
- In this lecture, we assumed oracle provides loop invariants
- Many different techniques for automating loop invariant generation; active research area
- Some other challenges: how to reason about the heap, concurrency, recursive functions ...
- Since program verification is undecidable, we can't always verify every correct program, but can verify many