Homework Assignment 3
Due Tuesday, February 20

The answers to the homework assignment should be your own individual work. Please hand in a hard copy of your solutions in class on the due date.

1. (20 points, 5 points each) Consider the following function and relation constants:

   - **isResearcher**(x): is a unary relation stating that person x is a researcher
   - **isReviewer**(x, y): is a binary relation stating that person x is a reviwer of paper y
   - **isAuthorOf**(x, y): is a binary relation stating that person x is an author of paper y
   - **areCoauthors**(x, y): is a binary relation stating that persons x and y are coauthors
   - **advisor**(x): is a unary function denoting person x's PhD advisor

   Give a translation from the following English sentences to the first-order language defined by the function and relation constants above.

   (a) Two people are coauthors if and only if they have written at least one paper together
   (b) Some researchers write all of their papers with at least one other researcher
   (c) If any person x coauthors a paper with another person y, then x can never review y's papers
   (d) Every researcher has written at least one paper with their PhD advisor

2. (15 points, 5 points each) Given binary relation constants p and q and a unary function constant f, consider the following structure defined by the universe of discourse $U = \{\star, \circ\}$ and the following interpretation I:

$$\begin{aligned} I(p) &= \{\langle \circ, \star \rangle, \langle \star, \circ \rangle\} \\ I(q) &= \{\langle \circ, \circ \rangle, \langle \circ, \star \rangle\} \\ I(f) &= \{\star \mapsto \circ, \circ \mapsto \circ\} \end{aligned}$$

   Under this structure and variable assignment $\sigma : [x \mapsto \star, y \mapsto \circ, z \mapsto \circ]$, state whether the following formulas evaluate to true or false. No partial credit will be given for wrong answers, and no explanation is necessary.

   (a) $p(x, y) \rightarrow (\forall z.\exists w.\ q(z, f(w)))$
   (b) $p(y, x) \rightarrow (\exists z.\forall w.\ q(z, w))$
   (c) $\forall x.\exists y.((p(x, y) \wedge q(y, x)) \rightarrow p(y, x))$

3. (30 points, 10 points each) For each of the following sentences, identify whether it is valid, satisfiable, or unsatisfiable. If the formula is valid or unsatisfiable, prove it using the semantic argument method. If the formula is satisfiable but not valid, provide (i) a structure that satisfies the formula, and (ii) a structure that falsifies the formula.

   (a) $(\exists x.p(x) \vee q(x)) \leftrightarrow \exists x.p(x) \vee \exists x.q(x)$

(b) $(\forall x, y.\ (p(x, y) \rightarrow p(y, x))) \rightarrow \forall z.p(z, z)$

(c) $(\forall x.(\neg p(x) \vee \neg q(x))) \leftrightarrow (\neg \forall x.(p(x) \wedge q(x)))$

(d) $\exists y.\forall x.p(y, x) \wedge \exists x.\forall y.\neg p(y, x)$

4. (30 points) In this problem, we will prove the undecidability of determining validity in FOL by reducing it to the **Post-Correspondence Problem (PCP)**, which is known to be undecidable. Given a finite sequence $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ where each s_i, t_i is a non-empty bitstring, the PCP problem is to determine whether there exists a sequence of indices $i_1, \ldots, i_n$ such that $s_{i_1} s_{i_2} \ldots s_{i_n}$ and $t_{i_1} t_{i_2} \ldots t_{i_n}$ are identical. For instance, for the sequence

$$(1, 101), (10, 00), (011, 11)$$

the index sequence $1, 3, 2, 3$ is a solution.

(a) (2 points) Let ϵ be an object constant representing an empty string, and let f_0, f_1 be unary function constants such that $f_0(s) = s0$ and $f_1(s) = s1$. Show the representation of the bitstring 011 using ϵ, f_0 and f_1.

(b) (12 points) Let $P(x, y)$ be a binary predicate that is true iff there exists some sequence of indices $i_1, \ldots, i_n$ such that x equals $s_{i_1} \ldots s_{i_n}$ and y equals $t_{i_1} \ldots t_{i_n}$. Provide a recursive definition of $P(x, y)$, using the abbreviation f_w to denote $f_{w_k} \circ \ldots \circ f_{w_1}$ for the bitstring $w = w_1 \ldots w_k$.

(c) (6 points) Given a PCP instance I, express the existence of a solution to I in terms of the binary predicate P defined above for this instance I.

(d) (5 points) Using your answers to parts (b) and (c) above, write a FOL formula ϕ such that ϕ is valid iff a given PCP instance $(s_1, t_1), \ldots, (s_k, t_k)$ has a solution.

(e) (5 points) Explain why your answer to part (d) entails the undecidability of the FOL validity problem under the assumption that PCP is undecidable.