Homework Assignment 5
Due Tuesday, March 27, 2018

1. (10 points) For each of the following claims, state whether it is true or false. Give a clear and concise explanation to justify your answer.
   (a) Consider a first order-theory T with signature Σ and axioms A. Let F_A denote the conjunction of every axiom in A, and let ϕ be a formula over Σ. If ϕ is satisfiable modulo T, then $F_A \rightarrow \phi$ is also satisfiable in standard first-order logic.
   (b) Consider a first order-theory T with signature Σ and axioms A. Let F_A denote the conjunction of every axiom in A, and let ϕ be a formula over Σ. If $F_A \rightarrow \phi$ is satisfiable in standard first-order logic, then ϕ is satisfiable modulo T.

2. (15 points) Consider two theories T_1 and T_2 such that $\Sigma_1 = \Sigma_2 = \Sigma$ and $A_1 \supseteq A_2$. Also, let F be a formula over signature Σ.
   (a) If F is satisfiable modulo T_2, is it also satisfiable modulo T_1? Prove your answer or give a counterexample.
   (b) If F is valid modulo T_2, is it also valid modulo T_1? Prove your answer or give a counterexample.
   (c) If T_2 is complete, what, if anything, can we say about the completeness of T_1? Justify your answer.

3. (10 points) Consider the formula $\forall x.(2 \cdot x - 3 > y)$ in the theory of integers $T_{\mathbb{Z}}$. Give a Presburger arithmetic encoding of this formula.

4. (10 points) Decide whether the formula below is valid. If it is valid or unsatisfiable, use the semantic argument method to prove your claim. If the formula is satisfiable but not valid, provide both a falsifying and satisfying structure.

$$a\langle i \triangleleft e\rangle[j] = e \;\rightarrow\; i = j \vee a[j] = e$$

5. (10 points) Apply the congruence closure algorithm to decide the satisfiability of the following $T_=$ formula:

$$f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$$

   Show all the equalities processed by the algorithm and the DAG data structure (with representatives) before and after processing each equality.