1. (10 points) For each of the following claims, state whether it is true or false. Give a clear and concise explanation to justify your answer.

(a) Consider a first order-theory $T$ with signature $\Sigma$ and axioms $A$. Let $F_A$ denote the conjunction of every axiom in $A$, and let $\phi$ be a formula over $\Sigma$. If $\phi$ is satisfiable modulo $T$, then $F_A \rightarrow \phi$ is also satisfiable in standard first-order logic.

(b) Consider a first order-theory $T$ with signature $\Sigma$ and axioms $A$. Let $F_A$ denote the conjunction of every axiom in $A$, and let $\phi$ be a formula over $\Sigma$. If $F_A \rightarrow \phi$ is satisfiable in standard first-order logic, then $\phi$ is satisfiable modulo $T$.

2. (10 points) Decide whether the formula below is valid. If it is valid or unsatisfiable, use the semantic argument method to prove your claim. If the formula is satisfiable but not valid, provide both a falsifying and satisfying structure.

$$a(i \triangleleft e)[j] = e \rightarrow i = j \lor a[j] = e$$

3. (10 points) Apply the congruence closure algorithm to decide the satisfiability of the following $T_\omega$ formula:

$$f(g(x)) = g(f(x)) \land f(g(f(y))) = x \land f(y) = x \land g(f(x)) \neq x$$

You solution should provide detail at the level of the examples from class.

4. (20 points) Solve the following linear program using Simplex:

Maximize : $x_1 + 3x_2$

Subject to :

$-x_1 + x_2 \leq -1$

$-2x_1 - 2x_2 \leq -6$

$-x_1 + 4x_2 \leq 2$

$x_1, x_2 \geq 0$

Show the initial slack form representation and the auxiliary linear program needed to obtain a feasible basic solution. Also, show each step of the Simplex algorithm after performing a pivot operation. Your answer should provide detail at the level of the examples done in class. Use Bland’s rule for pivot selection.

5. (10 points) Recall that the Omega Test uses Fourier-Motzkin variable elimination to compute the real shadow.

(a) Use the Fourier-Motzkin technique to eliminate variable $x_1$ from the following system:

$$-5x_1 - 5x_2 \leq -11$$

$$2x_1 - 2x_2 \leq 1$$

$$-x_1 + 3x_2 \leq 3$$

(b) Using your answer to part (a), what does the Omega test conclude about the existence of an integer solution to this system based on the real shadow?