Relaxation-Based Techniques Overview

Main idea behind relaxation-based techniques for solving $A\vec{x} \leq \vec{b}$

1. First, ignore integrality requirement
2. Use LP algorithm (e.g., Simplex) to obtain rational solution
3. No rational solution $\Rightarrow$ unsatisfiable
4. Rational solution also integer $\Rightarrow$ satisfiable
5. Add additional constraint to guide search for integer solution
6. Repeat until (3) or (4) holds

Popular Relaxation-Based Techniques

Existing relaxation-based techniques:
- Branch-and-bound (B&B): simplest relaxation-based technique
- Gomory cuts: uses reasoning performed by Simplex to infer valid inequalities
- Branch-and-cut: combines B&B and Gomory cuts
- Cuts-from-proofs: generates cuts from proofs of unsatisfiability

We'll only talk about B&B and Cuts-from-Proofs

Branch and Bound

- B&B is simplest relaxation-based technique
- Suppose Simplex yields fractional solution
- Means there exists some variable $x_i$ assigned to non-integer $f_i$
- B&B constructs two subproblems:
  $A\vec{x} \leq \vec{b} \land x_i \leq \lfloor f_i \rfloor$
  $A\vec{x} \leq \vec{b} \land x_i \geq \lceil f_i \rceil$
- Original problem has a solution iff either subproblem has integer solution. Why?
- Because there is no integer point in range $(\lfloor f_i \rfloor, \lceil f_i \rceil)$, so aren’t missing any integer points

An Example

Consider the system:

\[
\begin{align*}
2x + y &\leq 5 \\
-x &\leq -1 \\
-y &\leq 0
\end{align*}
\]

Suppose Simplex yields $(\frac{5}{2}, 0)$

B&B constructs subproblem with $x \geq 3$

This problem has no rational solution

B&B constructs subproblem with $x < 2$

All vertices integers, thus Simplex yields integer solution $\Rightarrow$ satisfiable

Problem with Branch and Bound

- B&B very simple, but in many cases, it does not terminate!
- Consider the following set of inequalities:

\[
\begin{align*}
-3x + 3y + z &\leq -1 \\
3x - 3y + z &\leq 2 \\
z &\geq 0
\end{align*}
\]

No integer points, but infinitely many rational solutions
Overview of Cuts-from-Proofs

Example Using Branch and Bound

- Suppose Simplex yields the solution
  \[(x, y, z) = \left(\frac{1}{3}, 0, 0\right)\]
  for the previous problem. Branch and bound constructs two subproblems with additional constraints \(x \leq 0\) and \(x \geq 1\). For the subproblem where \(x \geq 1\), we obtain a new solution
  \[(x, y, z) = \left(1, \frac{2}{3}, 0\right)\]

Limitation of Branch and Bound

- As example illustrates, B&B does not have termination guarantees
- It always terminates if feasible region is bounded
- If feasible region is unbounded and does not contain integer points, B&B does not terminate

Cuts-from-Proofs: Motivating Example

- Cuts-from-Proofs generalizes B&B
- Instead of branching around coordinate planes, it computes a custom plane with no integer points
  \[-3x + 3y + 3z = -1\]
- Then, we create two subproblems by branching around this plane
- Neither subproblem has integer solution, so cuts-from-proof immediately terminates

Some Polyhedral Terminology

Before talking about cuts-from-proofs, some new terminology

- An inequality \(\pi \bar{x} \leq c\) called valid inequality if it is satisfied by all points in polytope
- A face \(F\) is the set of points in the polytope satisfying \(\pi \bar{x} = c\) for some valid inequality \(\pi \bar{x} \leq c\)
- A vertex of polytope is a face of dimension 0
- A facet of polytope \(P\) is a face of dimension \(\text{dim}(P) - 1\)

Overview of Cuts-from-Proofs

1. Run Simplex, obtain a point \(v\) on polytope
2. Compute defining constraints of this point
   - Intersection \(F\) of defining constraints contains a face of the polytope
   - \(F\) does not have to be point; can be line, plane, hyperplane . . .
3. Determine if \(F\) contains integer solutions
   - can be done in polynomial time using Hermite normal forms
4. If \(F\) contains integer points, perform regular B&B
   - There might be an integer that lies on \(F\) and inside polytope
Overview of Cuts-from-Proofs, cont

5. If $F$ has no integers, compute proof of unsatisfiability
   - Proof of unsat is implied by face and does not contain integers
   - Can also be computed in poly-time using Hermite normal forms

6. Compute planes closest to and on either side proof of unsat containing integer points

7. Solve two new subproblems stipulating solution must lie on one side of these planes

8. Problem has integer solution if either of these subproblems has solution

Why is HNF Useful?

- We can convert every matrix $A$ to its corresponding Hermite Normal Form representation, written $HNF(A)$
- The HNF representation of $A$ is unique
- Can be computed in polynomial time

Defining Constraints of Vertex

- Suppose we run Simplex on $A\vec{x} \leq \vec{b}$ and obtain point $v$
- $A_1\vec{x} = b_1$ is defining constraint of $v$ if:
  1. $A_1\vec{x} \leq b_1$ is a row of $A\vec{x} \leq \vec{b}$
  2. $v$ satisfies $A_1\vec{x}' = b_1$
- Simplex always returns a point, but intersection of defining constraints does not have to be a point
- It can be line, plane, or $k$-dimensional hyperplane

Example

- Consider again the system:
  $$-3x + 3y + z \leq -1$$
  $$3x - 3y + z \leq 2$$
  $$z = 0$$
- Suppose Simplex yields $(\frac{1}{3}, 0, 0)$
- What are the defining constraints?
  $$-3x + 3y + z = -1$$ and $z = 0$
- Intersection of defining constraints is the face shown in red line

Checking Integer Solutions on Face

Next step in algorithm: Decide if intersection of defining constraints contains integer points

- To do this, we need matrix representation called Hermite Normal Form (HNF)
- An $m \times m$ matrix $H$ is in HNF if:
  1. $H$ is lower-triangular
  2. Diagonal entries are positive ($h_{ii} > 0$)
  3. $h_{ij} \leq 0$ and $|h_{ij}| < h_{ii}$ for $i > j$

HNF of Matrix

- We can convert every matrix $A$ to its corresponding Hermite Normal Form representation, written $HNF(A)$
- The HNF representation of $A$ is unique
- Can be computed in polynomial time

Why is HNF Useful?

- Theorem: The linear equality system $A\vec{x} = \vec{b}$ has solutions if and only if $H^{-1}\vec{b}$ has integer entries (where $H$ is $HNF(A)$)
- Use theorem to determine if intersection of defining constraints has integer solution
- Let $A'\vec{x} = \vec{b}'$ be defining constraints
- Compute the Hermite Normal Form $H$ of $A'$ and check if $H^{-1}\vec{b}'$ has only integers
### Example

- For the point $(\frac{1}{3}, 0, 0)$, defining constraints were $z = 0$ and $-3x + 3y + z = -1$.
- Written as a matrix $A'\vec{x} = \vec{b}':
  \[
  \begin{bmatrix}
  0 & 0 & 1 \\
  -3 & 3 & 1
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  -1
  \end{bmatrix}
  \]
- Hermite normal form $H$ of $A'$:
  \[
  \begin{bmatrix}
  1 & 0 \\
  -2 & 3
  \end{bmatrix}
  \]
- $H^{-1}\vec{b}$ is $\begin{bmatrix}
  0 \\
  -\frac{1}{3}
  \end{bmatrix}$
- Does intersection of defining constraints have integer solution? No

### Proof of Unsatisfiability

- A proof of unsatisfiability of $A'\vec{x} = \vec{b}'$ is a single linear equality $E$ such that:
  1. $E$ is implied by $A'\vec{x} = \vec{b}'$
  2. $E$ does not have integer solutions
- In what way does $E$ prove $A'\vec{x} = \vec{b}'$ is unsatisfiable?
- From condition (1), we have $A'\vec{x} = \vec{b}' \Rightarrow E$
- Contrapositive of this is: $\neg E \Rightarrow A'\vec{x} \neq \vec{b}'$
- Since $E$ is unsat, $\neg E$ is true; thus, $A'\vec{x} \neq \vec{b}'$ always true
- Hence, $E$ establishes unsatisfiability of $A'\vec{x} = \vec{b}'$

### Computing Proof of Unsatisfiability

- We can compute proofs of unsatisfiability using HNF
- Let $A'\vec{x} = \vec{b}'$ be a system with no integer solutions.
- Compute Hermite Normal Form $H$ of $A'$ and $H^{-1}$
- Now, consider $H^{-1}A'\vec{x} = H^{-1}\vec{b}'$:
  \[
  \begin{bmatrix}
  a_1 & \cdots & a_n \\
  \vdots & \ddots & \vdots \\
  a_{2n-1} & \cdots & a_{2n}
  \end{bmatrix}
  \begin{bmatrix}
  \vec{x} \\
  \vec{y}
  \end{bmatrix}
  =
  \begin{bmatrix}
  e_1 \\
  \vdots \\
  e_m
  \end{bmatrix}
  \]
- Proof of unsatisfiability of $A'\vec{x} = \vec{b}'$:
  \[
  a_1 dx_1 + a_2 dx_2 + \ldots + a_5 dx_5 = n
  \]

### Example, cont

- Result of multiplication:
  \[
  \begin{bmatrix}
  0 & 0 & 1 \\
  -1 & 1 & 1
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  -\frac{1}{3}
  \end{bmatrix}
  \]
- What is proof of unsatisfiability?
  \[-3x + 3y + 3z = -1\]
- This equality has no integer solutions because the GCD of coefficients does not evenly divide constant on RHS.
- Furthermore, it is implied by the defining constraints $-3x + 3y + z = -1$ and $z = 0$
Branching around the Proof

- Consider proof of unsatisfiability of the system $A'\vec{x} = \vec{b}$
  
  $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = c$

- To "branch around" this plane, need to compute the closest planes parallel to the proof of unsatisfiability containing integer points

- Let $g$ be the GCD of $a_1, a_2, \ldots, a_n$

- The closest parallel planes containing integers given by:
  
  $\frac{a_1}{g} x_1 + \frac{a_2}{g} x_2 + \ldots + \frac{a_n}{g} x_n = \left\lfloor \frac{c}{g} \right\rfloor$
  
  $\frac{a_1}{g} x_1 + \frac{a_2}{g} x_2 + \ldots + \frac{a_n}{g} x_n = \left\lceil \frac{c}{g} \right\rceil$

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Branching Around the Proof, cont.

- Thus, if solution is to the "left" of proof plane, must satisfy:
  
  $\frac{a_1}{g} x_1 + \frac{a_2}{g} x_2 + \ldots + \frac{a_n}{g} x_n \leq \left\lfloor \frac{c}{g} \right\rfloor \quad (1)$

- And if it is to the "right", it must satisfy:
  
  $\frac{a_1}{g} x_1 + \frac{a_2}{g} x_2 + \ldots + \frac{a_n}{g} x_n \geq \left\lceil \frac{c}{g} \right\rceil \quad (2)$

- Thus, we construct two subproblems where we conjoin $A'\vec{x} \leq \vec{b}$ with (1), and another where we conjoin it with (2)

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Example

- Defining constraints for $\left( \frac{1}{2}, 0, 0 \right)$:
  
  $\begin{bmatrix} -3 & 3 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

- Earlier, we computed proof of unsatisfiability as:
  
  $-3x + 3y + 3z = -1$

- What are the closest parallel planes containing integers?
  
  $-x + y + z = -1$
  
  $-x + y + z = 0$

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What About an Unsatisfiable Subset?

- Ok, we excluded intersection of defining constraints from search space

- But what if there is a subset of these defining constraints whose intersection has no integers?

- If so, we would like to exclude this higher-dimensional subspace from the search space.

- Another guarantee: If we add new inequalities in a certain order, we are also guaranteed to eventually find proof of unsatisfiability for this higher dimensional face!

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What Have We Achieved?

- Guarantee: By branching around proof of unsatisfiability, Simplex will never yield point with same defining constraints again!

- Proof: Suppose defining constraints in current iteration were $A'\vec{x} = \vec{b}$ with proof of unsatisfiability $E$

- Suppose we get a vertex $v$ with same defining constraints next

- Observation 1: By definition, $v$ must satisfy $A'\vec{x} = \vec{b}$

- Observation 2: $v$ cannot satisfy $E$ because it would violate the new inequalities we added

- But since $A'\vec{x} = \vec{b} \Rightarrow E$, this implies $v$ can't satisfy $A'\vec{x} = \vec{b}$
Experiments

- We compared performance of this new algorithm against four other tools solving qff linear inequalities
  1. CVC3: uses Omega Test
  2. Yices: uses branch-and-cut (B&B + Gomory cuts)
  3. Z3: uses branch-and-cut (B&B + Gomory cuts)
  4. MathSAT: uses Omega Test + branch-and-cut

Experiments

Number of variables vs. average running time.

Number of variables vs. percent of successful runs with a 1200 second time-out.

Maximum coefficient vs. average running time for 10 × 20 system

Cuts-from-Proofs Summary

- New relaxation-based technique for solving qff linear integer inequalities
- Computes custom planes to branch around for each problem
- These custom planes are generated by computing proof of unsatisfiability for defining constraints
- At least for some problems, this algorithm does significantly better than existing ones