Review

- To decide satisfiability in theory of rationals, convert to DNF and express each conjunct as a linear program:
  - Maximize $y$
  - Subject to:
    - $\land a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i \land \land a_{i1}x_1 + \ldots + a_{in}x_n + y \leq \beta_i$
- Clause is satisfiable iff optimal value is ... ?
- Determine optimal value using Simplex

Slack form and basic solution

- Precursor to Simplex is to bring LP to standard and then to slack form – what are these?
- Variables appearing on LHS called basic variables, RHS ones called non-basic variables
- Invariant: Only non-basic variables can appear in the objective function
- What is the basic solution for slack form?
- What is feasible basic solution?

Simplex Review

- Phase 0: Express linear program in slack form
- Phase I: Compute a feasible basic solution, if one exists
- Phase II: Optimize value of objective function

Simplex Algorithm Optimization Phase Overview

- Starting with a feasible basic solution, each iteration rewrites one slack form into an equivalent slack form
- This rewriting is similar to Gaussian elimination: involves pivot operations on matrix
- Geometrically, each iteration of Simplex "walks" from one vertex to an adjacent vertex until it reaches a local maximum
- By convexity, local optimum is global optimum; thus algorithm can safely stop when local maximum is reached

Simplex Algorithm Optimization Phase

- When rewriting one slack form to another, goal is to increase value of objective function associated with basic solution
- Recall: Objective function is $z = v + \sum_{x_j \in N} c_j x_j$
- How can we increase value of $z$?
- If there is a term $c_j x_j$ with positive $c_j$, we can increase value of $z$ by increasing $x_j$'s value, i.e., by making $x_j$ a basic variable
- What if there are no positive $c_j$'s?
- Then, we know we can’t increase value of $z$, thus we are done!
Simplex Algorithm Optimization Phase, cont
- Suppose we can increase objective value, i.e., there exists a term $c_j x_j$ with positive $c_j$
- We want to increase $x_j$’s value, but is there a limit on how much we can increase $x_j$?
- Consider equality $x_i = b_i - a_{ij} x_j - \ldots$
- Observe: If $a_{ij}$ is positive and we increase $x_j$ beyond $\frac{b_i}{a_{ij}}$, $x_i$ becomes negative and we violate constraints
- Thus, the amount by which we can increase $x_j$ is limited by the smallest $\frac{b_i}{a_{ij}}$ among all $i$’s
- If there is no positive coefficient $a_{ij}$, we can increase $x_j$ (and thus $z$) without limit $\Rightarrow$ optimal solution $= \infty$

Simplex Algorithm Optimization Phase, cont
- Thus, given term $c_j x_j$ with positive $c_j$ in objective function, we want to increase $x_j$ as much as possible
- To increase $x_j$ as much as possible, we find equality that most severely restricts how much we can increase $x_j$
- Equality that most severely restricts $x_j$ has following characteristics:
  1. $x_j$’s coefficient $a_{ij}$ is positive (otherwise doesn’t limit $x_j$)
  2. has smallest value of $\frac{b_i}{a_{ij}}$ (most severely restricting)

Simplex Algorithm Optimization Phase, cont
- Suppose equality with basic var. $x_i$ is most restrictive for $x_j$
- Swap roles of $x_i$ and $x_j$ by making $x_j$ basic and $x_i$ non-basic
- To do this, rewrite $x_j$ in terms of $x_i$ and plug this in to all other equations; this operation is called a pivot
- After performing this pivot operation, what is new value of $x_j$?
- Assuming $b_i$ is non-zero, we have increased the value of $x_j$ from 0 to $\frac{b_i}{a_{ij}}$
- Thus, after performing pivot we still have feasible solution but objective value is now greater

Simplex Optimization Phase Summary
- Pivot operation exchanges a basic variable with a non-basic variable to increase objective value of basic solution
- Simplex repeats this pivot operation until one of two conditions hold:
  1. All coefficients in objective function are negative $\Rightarrow$ optimal solution found
  2. There exists a non-basic variable $x_j$ with positive coefficient $c_j$ in objective function, but all coefficients $a_{ij}$ are negative $\Rightarrow$ optimal solution $= \infty$

Example
\[
\begin{align*}
  z &= 3x_1 + 2x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
\end{align*}
\]
- How can we increase value of objective function?
- Which equality restricts $x_1$ the most?
- Rewrite $x_1$ in terms of $x_6$:
  \[
  x_1 = 9 - \frac{1}{3}x_2 - \frac{1}{2}x_3 - \frac{1}{3}x_6
  \]

Example, cont
- Plug this in for $x_1$ in all other equations (i.e., pivot):
  \[
  \begin{align*}
  z &= 27 + \frac{3}{2}x_2 + \frac{3}{2}x_3 - \frac{3}{2}x_6 \\
  x_1 &= 9 - \frac{3}{2}x_2 - \frac{3}{2}x_3 - \frac{3}{4}x_6 \\
  x_4 &= 21 - 6x_2 - \frac{1}{2}x_3 + \frac{1}{4}x_6 \\
  x_5 &= 6 - \frac{3}{2}x_2 - 2x_3 + \frac{1}{2}x_6 \\
\end{align*}
  \]
- How can we increase value of $z$?
- Which equality restricts $x_3$ the most?
- What is $x_3$ in terms of $x_1$, $x_2$, $x_5$?
  \[
  x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{3}{4}x_5 + \frac{1}{8}x_6
  \]
Degenerate Problems and Termination

- If problem is not degenerate, Simplex guaranteed to terminate for any pivot selection strategy (b/c objective value increases)

  - **Bad news:** For degenerate problems, Simplex might not terminate
  - **Good news:** There are pivot selection strategies for which Simplex is always guaranteed to terminate, even for degenerate problems

  - One such strategy is Bland’s rule: If there are multiple variables with positive coefficients in objective function, always choose the variable with smallest index

  - **Example:** If \( z = 2x_1 + 5x_2 - 4x_3 \), Bland’s rule chooses \( x_1 \) as new basic variable since it has smallest index

Can the Objective Value Decrease?

- Let \( c_n \) be the objective value at \( n \)'th iteration of Simplex, and let \( c_{n+1} \) be the objective value at \( n + 1 \)'th iteration.

- **Is it possible that \( c_{n+1} < c_n \)?**

  - Consider objective function at \( n \)'th iteration: \( z = v + \Sigma c_jx_j \)
  - What is objective value at \( n \)'th iteration?
  - Suppose Simplex makes \( x_j \) basic variable in next iteration.
  - At \( n \)'th iteration, value of \( x_j \) was 0 (since \( x_j \) non-basic)
  - At \( n + 1 \)'th iteration, \( x_j \geq 0 \) because we don’t violate non-negativity constraints
  - Thus, Simplex never decreases value of the objective function!

Degenerate Problems

- Objective value can’t decrease; but can it stay the same?

  - **Example:** Suppose we make \( x_2 \) the new basic variable, and most constraining equality is:

    \[
x_1 = x_2 + 2x_3 + x_4 \]

  - \( x_2 \)'s old value was 0; what is its new value?

  - Thus, the objective value does not decrease, but does not increase either!

  - These kinds of problems where objective value can stay the same after pivoting are called degenerate problems

Simplex Algorithm Phases

- Simplex algorithm has two phases:

  1. Phase I: Compute a feasible basic solution, if one exists
  2. Phase II: Optimize value of objective function

- So far, we talked about the second phase, assuming we already have a feasible basic solution

- However, the initial basic solution might not feasible even if the linear program is feasible
### Example of Infeasible Initial Basic Solution

- Consider the following linear program:
  
  \[
  \begin{align*}
  z &= 2x_1 - x_2 \\
  x_3 &= 2 - 2x_1 + x_2 \\
  x_4 &= -x_1 + 5x_2
  \end{align*}
  \]

  - What is the initial basic solution?
  - Is this solution feasible?
  - Goal of Phase I of Simplex is to determine if a feasible basic solution exists, and if so, what it is.

### Overview of Phase I

- To find an initial basic solution, we construct an auxiliary linear program $L_{aux}$.
  - This auxiliary linear program has the property that we can find a feasible basic solution for it after at most one pivot operation.
  - Furthermore, original LP problem has a feasible solution if and only if the optimal objective value for $L_{aux}$ is zero.
  - If optimal value of $L_{aux}$ is 0, we can extract basic feasible solution of original problem from optimal solution to $L_{aux}$.

### Constructing the Auxiliary Linear Program

- Consider the original LP problem:
  
  \[
  \begin{align*}
  \text{Maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
  \text{Subject to:} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i \in [1,m]) \\
  & \quad x_j \geq 0 \quad (j \in [0,n])
  \end{align*}
  \]

  - This problem is feasible iff the following LP problem $L_{aux}$ has optimal value 0:
    
    \[
    \begin{align*}
    \text{Maximize} & \quad -x_0 \\
    \text{Subject to:} & \quad \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad (i \in [1,m]) \\
    & \quad x_j \geq 0 \quad (j \in [0,n])
    \end{align*}
    \]

### Justification for Auxiliary LP

- Suppose $x_0$ has optimal value 0. Then clearly $a_{ij} x_j \leq b_i$ is satisfied for all inequalities.

- Suppose original problem has feasible solution $z^*$. Then $z^*$ combined with $x_0 = 0$ is feasible solution for $L_{aux}$.

- Due to the non-negativity constraint, $-x_0$ can be at most 0; thus, this solution is optimal for $L_{aux}$.

### Finding Feasible Basic Solution for $L_{aux}$

- So far, we argued that original problem $L$ has feasible solution iff $L_{aux}$ has optimal value 0.

- But we still need to figure out how to find feasible basic solution to $L_{aux}$.

- **Next:** We’ll see how we can find feasible basic solution for $L_{aux}$ after one pivot operation.

### Auxiliary Problem in Slack Form

- \[
  \begin{align*}
  z &= -x_0 \\
  x_i &= b_i + x_0 - \sum_{j=1}^{n} a_{ij} x_j
  \end{align*}
  \]

  - If all $b_i$’s are positive, basic solution already feasible.

  - If there is at least some negative $b_i$, find equality $x_i$ with most negative $b_i$.

  - Make $x_i$ new basic variable, and $x_j$ non-basic.

  - **Claim:** After this one pivot operation, all $b_i$’s are non-negative; thus basic solution is feasible.
Why is This True?

- Suppose this equality has most negative $b_i$:
  \[ x_i = b_i + x_0 - \sum_{j=1}^{n} a_{ij} x_j \]

- Rewrite to make $x_0$ basic:
  \[ x_0 = -b_i + x_i + \sum_{j=1}^{n} a_{ij} x_j \]

- Now, $-b_i$ is positive and greater than all other $|b_j|$’s

- Thus, when we plug in equality for $x_0$ into other equations, their new constants will be positive

- Hence, we find a feasible basic solution after at most one pivot step

Example, cont

- After pivoting, we obtain the new slack form:
  \[
  \begin{align*}
  z &= -4 - z_1 - x_1 + 5x_2 \\
  x_3 &= 6 - z_1 - 4z_2 + x_4 \\
  x_0 &= 4 + x_1 + x_2 - 5x_2 
  \end{align*}
  \]

- What is current objective value?

- How can we increase it?

- Which equation constrains $x_2$ the most?

- Swap $x_2$ and $x_0$:
  \[ x_2 = \frac{4}{5} \cdot x_0 + x_4 + x_1 \]

Example

- Consider the following linear program from earlier:
  \[
  \begin{align*}
  z &= 2z_1 - x_2 \\
  x_3 &= 2 - 2z_1 + x_2 \\
  x_4 &= -4 - x_1 + 5x_2 
  \end{align*}
  \]

- Construct $L_{aux}$:
  \[
  \begin{align*}
  z &= -z_0 \\
  x_3 &= 2 + 2z_0 - 2z_1 + x_2 \\
  x_4 &= -4 + 2z_0 - z_1 + 5x_2 
  \end{align*}
  \]

- Which equation has most negative constant?

- Swap $x_4$ and $x_0$:
  \[ x_0 = 4 + x_4 + x_1 - 5x_2 \]

Example, cont

- After pivoting, new slack form:
  \[
  \begin{align*}
  z &= -z_0 \\
  x_2 &= \frac{4}{5} - \frac{x_1}{5} - \frac{x_3}{5} + \frac{x_4}{5} \\
  x_3 &= 1 + \frac{9}{5} - \frac{x_1}{5} + \frac{x_4}{5} 
  \end{align*}
  \]

- Objective function cannot be increased, so we are done!

- In original problem, objective function was $z = 2z_1 - x_2$

- Since $x_2$ is now a basic variable, substitute for $x_2$ with RHS:
  \[ z = \frac{-4}{5} + \frac{9z_1}{5} - \frac{x_3}{5} \]

- Thus, Phase I returns the following slack form to Phase II:
  \[
  \begin{align*}
  z &= \frac{-4}{5} + \frac{9z_1}{5} - \frac{x_3}{5} \\
  x_2 &= \frac{4}{5} - \frac{x_1}{5} + \frac{x_4}{5} \\
  x_3 &= 1 + \frac{9}{5} - \frac{x_1}{5} + \frac{x_4}{5} 
  \end{align*}
  \]

Summary

- To solve constraints in $T_\mathbb{Z}$ (linear inequalities over rationals), we use Simplex algorithm for LP

- Simplex has two phases

- In first phase, we construct slack form such that it has a basic feasible solution

- In second phase, we start with basic feasible solution and rewrite one slack form into equivalent one until objective value can’t increase

- Although Simplex is a worst-case exponential, it is more popular than polynomial-time algorithms for LP

Theory of Integers

- Earlier, we talked about the theory of integers $T_\mathbb{Z}$:
  \[ \Sigma_\mathbb{Z} : \{ \ldots, -2, -1, 0, 1, 2, \ldots, -3, -2, -1, 2, 3, \ldots, +, -, =, > \} \]

- Recall: Equally expressive as Presburger arithmetic

- Next two lectures: Look at algorithms for deciding satisfiability in quantifier-free fragment of $T_\mathbb{Z}$
As in previous two lectures, we’ll consider $T^*_Z$ formulas without disjunctions.

**Problem we want to solve:** Given an $m \times n$ matrix $A$ with only integer coefficients and a vector $\vec{b}$ in $\mathbb{Z}^n$, does $A\vec{x} \leq \vec{b}$ have any integer solutions?

**Note:** rational solutions not ok – only accept integers!

This requirement actually makes problem much harder.

Finding solution over rationals is poly-time, but integer problem is NP-complete even without disjunctions.

As before the system $A\vec{x} \leq \vec{b}$ defines a polytope.

Earlier, we asked the question: Is the polytope empty?

This time, we want to know if polytope contains integer points.

While the polytope is convex, the space formed by all integer points in polytope is not convex.

Unfortunately, non-convexity makes problem much harder to solve.

Consider the set of linear inequalities:

\[
3x + 3y \leq 2 \\
3x + 3y \geq 1
\]

This problem has rational-valued solutions, e.g., $x = \frac{1}{4}, y = \frac{1}{4}$.

But it doesn’t have integer solutions.

In general, if $A\vec{x} \leq \vec{b}$ has integer solutions, it also has rational solutions.

But if it has rational solutions, this does not imply it also has integer solutions.

Two different techniques for solving linear integer inequalities.

1. **Elimination-based techniques:** Omega Test, Cooper’s method
2. **Relaxation-based techniques:** Branch-and-bound, Gomory cuts, Cuts-from-Proofs

Elimination-based techniques eliminate variables one by one until system becomes trivially solvable.

Relaxation-based techniques first drop the integrality requirement and look for a real valued solution.

Then, they iteratively introduce additional constraints to guide search for integer solution.

Called “relaxation-based” because they first solve LP-relaxation.

Next lecture: Talk about an elimination-based technique called Omega test.

Next next lecture: Talk about two relaxation-based techniques:

1. Branch-and-Bound
2. Cuts-from-Proofs