CS389L: Automated Logical Reasoning

Lecture 6: First Order Logic
Syntax and Semantics

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Overview

- So far: Automated reasoning in propositional logic.
- Propositional logic is simple and easy to automate, but not very expressive
- Today: First order logic, also known as relational logic, predicate logic, or first-order predicate calculus
- Much richer and more expressive, but does not admit completely automated reasoning (more on this later)

The Plan

- Syntax and informal semantics (review, today's lecture)
- Formal semantics, model theory (today's lecture)
- Semantic argument method for FOL and properties (next lecture)
- Unification, clausal form (third lecture)
- Resolution and first-order theorem proving (fourth lecture)

Constants in First-Order Logic

- In propositional logic, we had two constants $\top$ and $\bot$
- In first order logic, three kinds of constants:
  1. object constants
  2. function constants
  3. relation constants

Object Constants

- Object constants refer to objects in a universe of discourse.
- Example: If our universe of discourse is people, object constants can be $jack, jane, joe, \ldots$
- As a convention, we will use letters starting with $a - t$ or digits to denote object constants.
- Example: a, *art*, *beth*, 1 etc. refer to object constants.

Function Constants

- Function constants refer to functions
- Examples: *motherOf*, *ageOf*, *plus*, *times*, ...
- Each function constant has an associated arity indicating its number of arguments
- Example: *mother* has arity 1 (unary), *times* has arity 2 (binary) etc.
- An object constant is really a special case of a function constant with arity 0

Relation Constants

- Relation constants refer to relations between or properties of objects
- Example: *loves*, *betterthan*, *ishappy*, ...
- Each relation constant also has an associated arity
- Example: *loves* has arity 2, *ishappy* has arity 1 etc.

Terms

- A set of object, function, and relation constants C, F, R specifies a first-order language, written $L(C, F, R)$
- C, F, R form the signature of the language.
- Terms t for a first order language are generated using C, F
- Basic terms: Any object constant in C and variables, denoted $x, y, z, \dots$
- Composite terms: $f(t_1, \dots, t_k)$ where $f \in F$ is function of arity k, and $t_1, \dots t_k$ are terms
- Examples: $mary$, x, $sister(mary)$, $price(x, macys)$, $age(mother(y))$, ...

Formulas

- Formulas of $L(C, F, R)$ are formed using the terms of this language, relation constants R, logical connectives $\neg, \wedge, \vee, \dots$, and quantifiers $\forall, \exists$.
- Atomic formula (predicate): Expression $p(t_1, \dots, t_n)$ where $p \in R$ (of arity n), and $t_1, \dots, t_n$ are terms of $L(C, F, R)$
- If F_1 and F_2 are formulas, then so is $F_1 \star F_2$ where $\star$ is any binary connective
- If F is a formula, then so is $\neg F$
- If F is a formula and x a variable, so are $\forall x.F$ (asserts facts about *all* objects) and $\exists x.F$ (asserts facts about *some* object)

Important Reminder

- Predicates (e.g., $p(x)$) and function terms (e.g., $f(x)$) look similar, but they are very different!
- Function terms can be nested within each other and inside relation constants: $f(f(x)), p(f(x)), \dots$
- Predicates such as $p(x)$ cannot be nested within function terms or other predicates!
- $f(p(x))$, $p(p(x))$ etc. not valid in FOL!

Quantifiers and Scoping

- The subformula embedded inside a quantifier is called the scope of that quantifier.
- Example: $\forall y.((\forall x.p(x)) \rightarrow q(x, y))$
- An occurrence of a variable is bound if it is in the scope of some quantifier.
- An occurrence of a variable is free if it is not in the scope of any quantifiers.

Free vs. Bound Variable Example

- Consider the formula:

$$\forall y.((\forall x.p(x)) \rightarrow q(x, y))$$

- Is variable y bound or free?
- Is first occurrence of x bound or free?
- What about second occurrence of x?

Closed, Open, and Ground Formulas

- A formula with no free variables is called a closed formula.
- A closed formula is also called a sentence.
- A formula containing free variables is said to be open.
- Example: Is the formula $\forall y.((\forall x.p(x)) \rightarrow q(x,y)$ closed or open?
- Is the formula $\forall y.((\forall x.p(x)) \rightarrow (\exists x.q(x,y)))$ a sentence?
- A formula is said to be ground if it contains no variables.
- Example: $p(a, f(b)) \rightarrow q(c)$ is ground.
- Is $\forall x.p(x)$ ground?

FOL example #1

- Fermat's Last Theorem: No three positive integers x, y, z satisfy the equation $x^n + y^n = z^n$ for any integer n greater than 2.
- Assuming universe is integers, how do we express this theorem in FOL using function constant ˆ and relation constants $>, =$?

FOL example #2

- Consider the axiom schema of unrestricted comprehension in naive set theory:

  "There exists a set whose members are precisely those objects that satisfy predicate P"

- Using predicates $IsSet, \in, P$, express this in FOL

FOL Example #3

- Consider the statement "CS389L is taken only by those students who do not take CS388L"
- Express thise sentence in FOL using binary relation constant $takes$, and unary relation constant $student$

One Last Example

- Given binary relation friend, how do we say this in FOL?
- *"Every pair of friends has something in common"*
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Semantics of First Order Logic

- In propositional logic, the concepts of interpretation, satisfiability, validity were all straightforward.
- In FOL, these concepts are a bit more involved ...
- To give semantics to FOL, we need to talk about a universe of discourse (also sometimes called just "universe" or "domain")

Universe of Discourse

- A universe of discourse is a *non-empty* set of objects about which we want to say something
- Universe of discourse can be finite, countably infinite, or uncountably infinite, but not empty
- Example universes:
  - Set of non-negative integers
  - Set of real numbers
  - The set of suits in playing cards
  - Students in this class

First-Order Interpretations

- An interpretation for a first order language $L(C, F, R)$ is a mapping I from C, F, R to objects in universe U
- I maps every $c \in C$ to some member of U: $I(c) \in U$
- I maps every n-ary function constant $f \in F$ to an n-ary function $f^I : U^n \to U$
- I maps every n-ary relation constant $p \in R$ to an n-ary relation p^I such that $p^I \subseteq U^n$
- Observe: A first-order interpretation does not talk about variables (only constants)

An Example

- Consider the first order language containing object constants $\{a, b, c\}$, unary function constant f, and ternary relation constant r.
- Universe of discourse: $U = \{1, 2, 3\}$
- Possible interpretation I:

$$I(a) = 1, I(b) = 2, I(c) = 2$$

$$I(f) = \{1 \to 2, 2 \to 3, 3 \to 3\}$$

$$I(r) = \{\langle 1, 2, 1\rangle, \langle 2, 2, 1\rangle\}$$

Another Example

- Consider the universe of discourse $U = \{\square, \triangle\}$
- For an object constant a, what are the possible interpretations?
- For a unary function constant f, what are the possible interpretations?
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- For a unary relation constant r, what are all the possible interpretations?
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First-Order Structures and Variable Assignments

- A structure $S = \langle U, I\rangle$ for a first order language consists of a universe of discourse of U and an interpretation I.
- This is sometimes also called an algebra.
- A variable assignment σ (or assignment) to a FOL formula ϕ in a structure $S = \langle U, I\rangle$ is a mapping from variables in ϕ to an element of U.
- Example: Given $U = \{\square, \triangle\}$, a possible variable assignment for x: $\sigma(x) = \triangle$

Evaluation of Terms

- We define how to evaluate a term t under interpretation I and assignment σ, written $\langle I, \sigma\rangle(t)$
- Object constants: $\langle I, \sigma\rangle(a) = I(a)$
- Variable terms: $\langle I, \sigma\rangle(v) = \sigma(v)$
- Function terms:

$$\langle I, \sigma\rangle(f(t_1, \ldots, t_k)) = I(f)(\langle I, \sigma\rangle(t_1), \ldots, \langle I, \sigma\rangle(t_k))$$

Example: Evaluation of Terms

- Consider a first-order language containing object constants a, b and binary function f
- Consider universe $\{1, 2\}$ and interpretation I:

$$I(a) = 1 \quad I(b) = 2$$
$$I(f) = \{\langle 1,1\rangle \mapsto 2, \langle 1,2\rangle \mapsto 2, \langle 2,1\rangle \mapsto 1, \langle 2,2\rangle \mapsto 1\}$$

- Consider variable assignment $\sigma : \{x \mapsto 2, y \mapsto 1\}$
- Under I and σ, what do these terms evaluate to?

$$\begin{array}{rcl} f(a,y) & = & \\ f(x,b) & = & \\ f(f(x,b), f(a,y)) & = & \end{array}$$

Evaluation of Formulas, Notation

- We define evaluation of formula F under structure $S = \langle U, I\rangle$ and variable assignment σ.
- If F evaluates to true under U, I, σ, we write $U, I, \sigma \models F$
- If F evaluates to false under U, I, σ, we write $U, I, \sigma \not\models F$
- We define the semantics of $\models$ inductively.

Evaluation of Formulas, Bases Cases

- Base case I:

$$U, I, \sigma \models \top \qquad U, I, \sigma \not\models \bot$$

- Base case II:

$$U, I, \sigma \models p(t_1, \ldots, t_k) \;\text{ iff }\; \langle\, \langle I, \sigma\rangle(t_1), \ldots, \langle I, \sigma\rangle(t_k)\,\rangle \in I(p)$$

Example I: Evaluation of Formulas

- Consider a first-order language containing object constants a, b and unary function f, and binary relation constant p
- Consider universe $\{\star, \bullet\}$ and interpretation I:

$$I(a) = \star \quad I(b) = \bullet$$
$$I(f) = \{\star \mapsto \bullet, \bullet \mapsto \star\}$$
$$I(p) = \{\langle \bullet, \star\rangle, \langle \bullet, \bullet\rangle\}$$

- Consider variable assignment $\sigma : \{x \mapsto \star\}$
- Under U, I and σ, what do these formulas evaluate to?

$$\begin{array}{rcl} p(f(b), f(x)) & = & \\ p(f(x), f(b)) & = & \\ p(a, f(x)) & = & \end{array}$$

Evaluation of Formulas II

- Inductive semantics for boolean connectives:

$$\begin{array}{ll} U, I, \sigma \models \neg F & \text{iff } U, I, \sigma \not\models F \\ U, I, \sigma \models F_1 \wedge F_2 & \text{iff } U, I, \sigma \models F_1 \text{ and } U, I, \sigma \models F_2 \\ U, I, \sigma \models F_1 \vee F_2 & \text{iff } I, \sigma \models F_1 \text{ or } U, I, \sigma \models F_2 \\ U, I, \sigma \models F_1 \rightarrow F_2 & \text{iff, } U, I, \sigma \not\models F_1 \text{ or } U, I, \sigma \models F_2 \\ U, I, \sigma \models F_1 \leftrightarrow F_2 & \text{iff, } \;U, I, \sigma \models F_1 \text{ and } U, I, \sigma \models F_2, \\ & \quad \text{or } U, I, \sigma \not\models F_1 \text{ and } I, \sigma \not\models F_2 \end{array}$$

Variant of Variable Assignment

- We still need to evaluate formulas containing quantifiers!
- But to do that, we first define an x-variant of a variable assignment.
- An x-variant of assignment σ, written $\sigma[x \mapsto c]$, is the assignment that agrees with σ for assignments to all variables except x and assigns x to c.
- Example: If $\sigma : \{x \mapsto 1, y \mapsto 2\}$, what is $\sigma[x \mapsto 3]$?

Evaluation of Formulas II

- We can now give semantics to quantifiers:
- Universal quantifier:

$$U, I, \sigma \models \forall x.F \quad \text{iff} \quad \text{for all } o \in U,\ U, I, \sigma[x \mapsto o] \models F$$

- Existential quantifier:

$$U, I, \sigma \models \exists x.F \quad \text{iff} \quad \text{there exists } o \in U \text{ s.t } U, I, \sigma[x \mapsto o] \models F$$

Example III: Evaluation of Formulas

- Consider universe $\{\star, \bullet\}$, variable assignment $\sigma : \{x \mapsto \star\}$, and interpretation I:

$$I(a) = \star \quad I(b) = \bullet$$
$$I(f) = \{\star \mapsto \bullet, \bullet \mapsto \star\}$$
$$I(p) = \{\langle \bullet, \star \rangle, \langle \bullet, \bullet \rangle\}$$

- Under U, I and σ, what do these formulas evaluate to?

$$\forall x.p(x, a) \quad =$$
$$\forall x.p(b, x) \quad =$$
$$\exists x.p(a, x) \quad =$$
$$\forall x.\ (p(a, x) \to p(b, x)) \quad =$$
$$\exists x.\ (p(f(x), f(x)) \to p(x, x)) \quad =$$

Satisfiability and Validity of First-Order Formulas

- A first-order formula F is satisfiable iff there exists a structure S and variable assignment σ such that

$$S, \sigma \models F$$

- Otherwise, F is unsatisfiable.
- A structure S is a model of F, written $S \models F$, if for all variable assignments $\sigma \in X \to U$, $S, \sigma \models F$.
- A first-order formula F is valid, written $\models F$ if for all structures S, $S, \sigma \models F$

Satisfiability and Validity Examples

- Is the formula $\forall x.\exists y.p(x, y)$ satisfiable?
- Satisfying interpretation:
- Is this formula valid?
- Falsifying interpretation:
- Is the formula $\forall x.(p(x, x) \to \exists y.p(x, y))$ valid?
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More Satisfiability and Validity Examples

- Is the formula $(\exists x.p(x)) \to p(x)$ sat, unsat, or valid?
- Satisfying U, I, σ:
- Falsifying interpretation:
- Is the formula $(\forall x.p(x)) \to p(x)$ sat, unsat, or valid?
- What about $(\forall x.(p(x) \to q(x))) \to (\exists x.(p(x) \land q(x)))$?
- Satisfying interpretation:
- Falsifying interpretation:

Understanding Models

- Recall: A structure S is a model of a formula if for all σ, $S, \sigma \models F$
- Consider a formula F such that $S, \sigma \models F$. Is S a model F?
- Consider a sentence F such that $S, \sigma \models F$. Is S a model F?
- Consider a ground formula F such that $S, \sigma \models F$. Is S a model of F?

Summary

- Today: Syntax and formal semantics of FOL
- Next lecture:
  - Semantic argument method for FOL
  - Properties of first-order logic: decidability results, compactness