

## Problem Set 5

The answers to the homework assignment should be your own individual work. Please hand in a hard copy of your solutions in class on the due date.

1. (10 points) For each of the following claims, state whether it is true or false. Give a clear and concise explanation to justify your answer.
  - (a) Consider a first order-theory  $T$  with signature  $\Sigma$  and axioms  $A$ . Let  $F_A$  denote the conjunction of every axiom in  $A$ , and let  $\phi$  be a formula over  $\Sigma$ . If  $\phi$  is satisfiable modulo  $T$ , then  $F_A \rightarrow \phi$  is also satisfiable in standard first-order logic.
  - (b) Consider a first order-theory  $T$  with signature  $\Sigma$  and axioms  $A$ . Let  $F_A$  denote the conjunction of every axiom in  $A$ , and let  $\phi$  be a formula over  $\Sigma$ . If  $F_A \rightarrow \phi$  is satisfiable in standard first-order logic, then  $\phi$  is satisfiable modulo  $T$ .
2. (15 points) Consider two theories  $T_1$  and  $T_2$  such that  $\Sigma_1 = \Sigma_2 = \Sigma$  and  $A_1 \supseteq A_2$ . Also, let  $F$  be a formula over signature  $\Sigma$ .
  - (a) If  $F$  is satisfiable modulo  $T_2$ , is it also satisfiable modulo  $T_1$ ? Prove your answer or give a counterexample.
  - (b) If  $F$  is valid modulo  $T_2$ , is it also valid modulo  $T_1$ ? Prove your answer or give a counterexample.
  - (c) If  $T_2$  is complete, what, if anything, can we say about the completeness of  $T_1$ ? Justify your answer.
3. (10 points) Consider the formula  $\forall x. (2 \cdot x - 3 > y)$  in the theory of integers  $T_{\mathbb{Z}}$ . Give a Presburger arithmetic encoding of this formula.
4. (10 points) Decide whether the formula below (over the theory of arrays) is valid. If it is valid or unsatisfiable, use the semantic argument method to prove your claim. If the formula is satisfiable but not valid, provide both a falsifying and satisfying structure.

$$a\langle i \triangleleft e \rangle[j] = e \rightarrow i = j \vee a[j] = e$$

5. (10 points) Apply the congruence closure algorithm to decide the satisfiability of the following  $T_{=}$  formula:

$$f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$$

Your solution should provide detail at the level provided in lecture.