Problem Set 8 Solutions

1. Recall that the Nelson-Oppen method requires the theories to be combined to be *stably infinite*. Give an example to demonstrate why this restriction is necessary. Specifically, give two theories $T_1, T_2,$ and a $(T_1 \cup T_2)$-formula $F$ such that $F$ is unsatisfiable but we would conclude otherwise using the Nelson-Oppen method.

2. Consider the following formula $F$ in $T_\mathbb{R} \cup T_\mathbb{Z}$:

$$g(f(x - 2)) = x + 2 \land g(f(y)) = y - 2 \land y = x - 2$$

(a) Purify $F$ by writing it as an equisatisfiable formula of the form $F_1 \land F_2$ such that $F_1$ is in $T_\mathbb{R}$ and $F_2$ is in $T_\mathbb{Z}$.

(b) Decide the satisfiability of $F$ using the Nelson-Oppen method.

3. Recall that the DPLL($T$) framework invokes the solver for theory $T$ to learn conflict clauses and theory propagation lemmas. In this question, we will explore the inference of theory propagation lemmas.

   (a) Let $D_T$ be a decision procedure for a conjunction of $\Sigma_T$ literals. Describe how one can perform exhaustive theory propagation for $T$ without modifying the decision procedure $D_T$. (Note: *exhaustive* means that we want to infer all theory propagation lemmas.)

   (b) Suppose $T$ is the theory of equality with uninterpreted functions. Explain how we can infer theory propagation lemmas for this theory after running the congruence closure algorithm on the current partial assignment without rerunning the congruence closure algorithm.

4. $\phi^+(L)$ denotes formula $\phi$ with leaf $L$ replaced by $\top$. A leaf is a specific occurrence of a subformula that only contains predicates, functions, and negation (and no other boolean connectives). For instance, $a = b \land \neg(a = b) \land a = b$ has three leaves: $\neg(a = b)$ and two separate leaves $a = b$. Prove that $\phi \Rightarrow \phi^+(L)$ where $\phi$ is an SMT formula in NNF and $L$ is a leaf.