CS311H: Discrete Mathematics

Recurrence Relations

Instructor: Işıl Dillig

Recall: Recursively Defined Sequences

- In previous lectures, we looked at recursively-defined sequences
- Example: What sequence is this?

$$a_0 = 1$$
$$a_n = a_{n-1} + 1$$

- Another example: Fibonacci numbers

$$f_0 = 1$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

Recurrence Relations

- Recurively defined sequences are often referred to as recurrence relations
- The base cases in the recursive definition are called initial values of the recurrence relation
- Example: Write recurrence relation representing number of bacteria in n'th hour if colony starts with 5 bacteria and doubles every hour?

Closed Form Solutions

- Recurrence relations are often very natural to define, but we usually need to find closed form solution
- Recall: Closed form solution defines n'th number in the sequence as a function of n
- What is closed form solution to the following recurrence?

$$a_0 = 0$$
$$a_n = a_{n-1} + n$$

Closed Form Solutions of Recurrence Relations

- Given an arbitrary recurrence relation, is there a mechanical way to obtain the closed form solution?
- Not for arbitrary, but for a subclass of recurrence relations
- A linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  where each c_i is a constant and c_k is non-zero
- The value of k is called the degree of the recurrence relation
- The recurrence relation for Fibonacci numbers is a degree 2 linear homogeneous recurrence

Examples and Non-Examples

- Which of these are linear homogenous recurrence relations with constant coefficients?
  - $a_n = a_{n-1} + 2a_{n-5}$
  - $a_n = 2a_{n-2} + 5$
  - $a_n = a_{n-1} + n$
  - $a_n = a_{n-1} \cdot a_{n-2}$
  - $a_n = n \cdot a_{n-1}$

Characteristic Polynomial

- Cook-book recipe for solving linear homogenous recurrence relations with constant coefficients
- Definition: The characteristic equation of a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$ is

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$$

- i.e., Replace a_{n-i} with $r^{n-i-(n-k)}$

Characteristic Equation Examples

- What are the characteristic equations for the following recurrence relations?
  - $f_n = f_{n-1} + f_{n-2}$
  - $a_n = 2a_{n-1}$
  - $a_n = 2a_{n-1} + 5a_{n-3}$

Characteristic Roots

- The characteristic roots of a linear homogeneous recurrence relation are the roots of its characteristic equation.
- What are the characteristic roots of the following recurrence relations?
  - $a_n = 2a_{n-1} + 3a_{n-2}$
  - $f_n = f_{n-1} + f_{n-2}$

Theorem I for Solving Linear Homogenous Recurrence Relations

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with k distinct characteristic roots $r_1, \ldots, r_k$.

- Then the closed form solution for a_n is of the form:

$$\alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

- Furthermore, given k initial conditions, the constants $\alpha_1, \ldots, \alpha_k$ are uniquely determined
- Note: Won't do the proof because requires a good amount of linear algebra

Example

Find a closed form solution for the recurrence $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$

- Characteristic equation:
- Characteristic roots:
- Coefficients:
- Closed-form solution:

Generalized Theorem

- So far, we assume all characteristic roots are distinct – what happens if this is not the case?
- Theorem: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with t distinct characteristic roots $r_1, \ldots, r_k$ with multiplicities $m_1, \ldots, m_k$. Then solutions are of the form:

$$a_n = \sum_{i=0}^{t} (\alpha_{i,0} + \alpha_{i,1} \cdot n + \ldots + \alpha_{i,m_{i-1}} \cdot n^{m_i - 1}) r_i^n$$

An Example

- Find closed form of recurrence $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ with initial conditions $a_0 = 1, a_1 = 3, a_2 = 7$
- Characteristic equation:
- Characteristic roots:
- Solution form:
- Coefficients:

Bitstring Example

- Recurrence relations useful for solving counting problems – how many bitstrings of length n without two consecutive 0's?
- Let a_n denote the number of bitstrings that do not contain two consecutive 0's.
- Write a recurrence relation for a_n:
- Initial conditions:
- Question: Find closed form solution for a_n
- Characteristic roots:

Solving Linear Non-Homogeneous Recurrence Relations

- How do we solve linear, but non-homogeneous recurrence relations, such as $a_n = 2a_{n-1} + 1$?
- A linear non-homogeneous recurrence relation with constant coefficients is of the form:

$$a_n = c_1 a_{n-1} + a_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

- The recurrence obtained by dropping $F(n)$ is called the associated homogeneous recurrence relation

Particular Solution

- A particular solution for a recurrence relation is one that satisfies the recurrence but not necessarily the initial conditions
- Example: Consider the recurrence $a_n = a_{n-1} + 1$ with initial condition $a_0 = 5$
- A particular solution for this recurrence is $a_n = n$, but it does not satisfy the initial condition

Theorem about Linear Non-homogeneous Recurrences

Suppose $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$ has particular solution a_n^p, and a_n^h is solution for associated homogeneous recurrence. Then every solution is of the form $a_n^p + a_n^h$.

Why is this theorem useful?

- If we can find a particular solution, then we can also mechanically find a solution that satisfies initial conditions.
- Example: Solve the recurrence relation $a_n = 3a_{n-1} + 2n$ with initial condition $a_1 = 3$
- A particular solution: $-n - \frac{3}{2}$ (Why?)
- Solutions for homogeneous recurrence:
- Solutions for recurrence:
- Solve for α:

How do we find a particular solution?

Theorem: Consider $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$ where:

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

- **Case 1:** If s is not a root of the associated characteristic equation, then there exists a particular solution of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

- **Case 2:** If s is a root with multiplicity m of the characteristic equation, then there exists a solution of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

Example I

- Consider again the recurrence $a_n = 3a_{n-1} + 2n$
- Here, $s = 1$ and characteristic root is 3
- Hence, there exists a particular solution of the form $p_1 n + p_0$
- Now, solve for p_0, p_1:

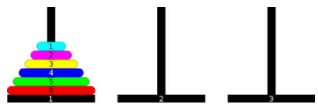
$$p_1 n + p_0 = 3(p_1(n-1) + p_0) + 2n$$

- Rearrange: $2n(p_1 + 1) + (2p_0 - 3p_1) = 0$
- A solution $p_1 = -1, p_0 = -\frac{3}{2}$
- A particular solution: $-n - \frac{3}{2}$

Example II

- Find a particular solution for $a_n = 6a_{n-1} - 9a_{n-2} + 2^n$
- Characteristic root:
- Particular solution of the form:
- Find p_0 such that $p_0 \cdot 2^n = 6(p_0 \cdot 2^{n-1}) - 9(p_0 \cdot 2^{n-2}) + 2^n$
- Solve for p_0:
- Particular solution:

Towers of Hanoi



- Given 3 pegs where first peg contains n disks
- **Goal:** Move all the disks to a different peg (e.g., second one)
- **Rule 1:** Larger disks cannot rest on top of smaller disks
- **Rule 2:** Can only move the top-most disk at a time
- **Question:** How many steps does it take to move all n disks?

A Recursive Solution

- Solve recursively – T_n is number of steps to move n disks
- **Base case:** $n = 1$, move disk from first peg to second: $T_1 = 1$
- **Induction:** Suppose we can move $n-1$ disks in T_{n-1} steps; how many steps does it take to move T_n disks?
- **Idea:** First move the topmost $n-1$ disks to peg 3; can be done in T_{n-1} steps
- Now, move bottom-most disk to peg 2 – takes just 1 step
- Finally, recursively move $n-1$ disks in peg 3 to peg 2 – can be done in T_{n-1} steps

Towers of Hanoi, cont.

- Recurrence relation:
- Initial condition:
- Now find closed form for T_n
- What is a particular solution?
- Solution for homogeneous recurrence:
- Solve for α:
- Solution for recurrence: