Coverset Induction with Partiality and Subsorts: A Powerlist Case Study

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Presented at ITP 2010

Overview

Motivation

Prove properties about recursive data parallel algorithms in a natural way using an interactive theorem prover.

Algorithms include:

- Parallel prefix sum
- Variable-size addition circuits
- Fast Fourier Transform and inverse
- Batcher sort

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Our Approach

Main design decisions:

- Formalize algorithms using Misra's Powerlists.
- Use Maude Inductive Theorem Prover (ITP) for verification.
  - Added generalized form of coverset induction to Maude ITP.
  - Coverset induction generates induction schemes from recursive terminating equations.
  - Reflective architecture that uses Maude's rewriter for simplification in theorems.

Powerlists

- Powerlists (Misra 94) are an algebraic type designed specifically for formalizing parallel data-recursive algorithms.
- Existing work by Misra, Kornerup, and Adams uses powerlists to formalize many different data parallel algorithms to be elegantly defined.
- Results have been shown with ACL2 and RRL, but existing automated proofs lack the elegance of the hand proofs.
- Our work is an effort to recapture the elegance within an inductive theorem prover.

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Powerlist definition

- Powerlists are lists with length 2^n for any $n \in \mathbb{N}$.
- Constructed in several ways:
  - Single elements $[e]$ of an arbitrary type,
  - Concatenating two similar powerlists p tie q, or
  - Interleaving two similar powerlists p zip q.
  - Nesting a powerlist as an element inside another $\langle p \rangle$.

  Powerlists p and q are similar $(\mathrm{sim?}(p, q))$ if they have the same number of nestings and the same number of elements at each nested level.
- In this presentation, we will only consider powerlists without nesting.

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Examples

$$[0\,1\,2\,3]\,\texttt{tie}\,[4\,5\,6\,7] = [0\,1\,2\,3\,4\,5\,6\,7]$$

$$[0\,1\,2\,3]\,\texttt{zip}\,[4\,5\,6\,7] = [0\,4\,1\,5\,2\,6\,3\,7]$$

Membership Equational Logic (MEL)

- Membership Equational Logic is a Horn logic with two levels of typing:

  Kinds Defined by the signature

  Sorts More refined mechanism defined by memberships in the theory.

  Maude statically type-checks at the kind level, while sorts are interpreted as predicates at runtime.
- The kind of a sort s is denoted by $[s]$.
- Sort and function definitions may be mutually recursive.

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Powerlists over Natural Numbers in Maude

```
fmod POWERLIST is protecting NAT .
  sort Pow .
  op [_] : Nat -> Pow [ctor].
  op _tie_ : [Pow] [Pow] -> [Pow] .

  vars M N : Nat .     vars P Q R S : Pow .

  cmb P tie Q : Pow if sim?(P, Q) = true .

  op sim? : Pow Pow -> Bool [comm].
  eq sim?([M], [N]) = true .
  eq sim?(P tie Q, [N]) = false .
  eq sim?(P tie Q, R tie S) = sim?(P, R) .

  op _zip_ : [Pow] [Pow] -> [Pow] .
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  eq (P tie Q) zip (R tie S) = (P zip R) tie (Q zip S) .
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Syntactic Sugar

This definition uses syntactic sugar:

```
op [_] : Nat -> Pow [ctor] .

op sim? : Pow Pow -> Bool [comm].

eq sim?(P tie Q, R tie S) = sim?(P, R) .
```

Semantically, these resolve into:

```
op [_] : [Nat] -> [Pow] .
cmb [N] : Pow if N : Nat.

op sim? : [Pow] [Pow] -> [Bool] [comm].
cmb sim?(P, Q) : Bool if P : Pow /\ Q : Pow .

ceq sim?(P tie Q, R tie S) = sim?(P, R)
 if P : Pow /\ Q : Pow /\ R : Pow /\ S : Pow .
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Prefix Sums

Given an associative binary operation $+$ over scalars,
the prefix sum $\mathrm{ps}(p)$ of a powerlist p over $+$ is:

$$\mathrm{ps}([x_0, x_1, \ldots, x_{n-1}]) = [x_0, x_0 + x_1, \ldots, x_0 + x_1 + \cdots + x_{n-1}]$$

Sequential Prefix Sums

Defining prefix sums in Maude.

```
--- Return prefix sum of list.
op ps : Pow -> Pow .
eq ps([ N ]) = [ N ] .
eq ps(P tie Q) = ps(P) tie (last(ps(P)) + ps(Q)) .

--- Return last element in list.
op last : Pow -> Nat .
eq last([N]) = N .  eq last(P tie Q) = last(Q) .

--- Add value to each element in Powerlist.
op _+_ : Nat Pow -> Pow .
eq M + [ N ] = [ M + N ] .
eq M + (P tie Q) = (M + P) tie (M + Q) .
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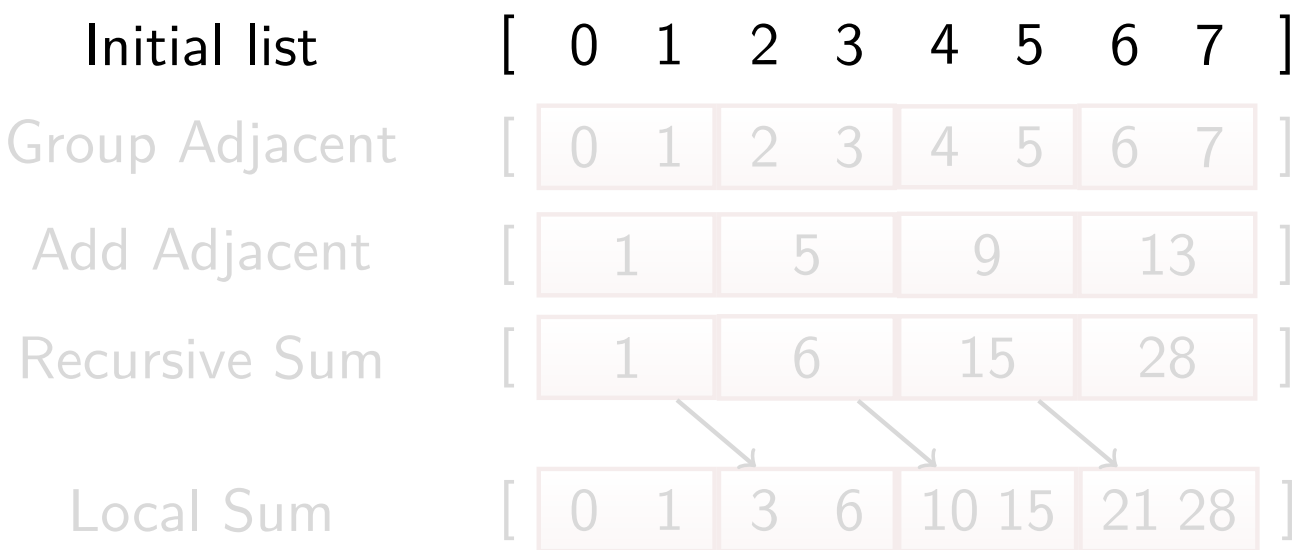
Ladner and Fischer Prefix Sums

Ladner and Fischer's approach operates on adjacent elements.



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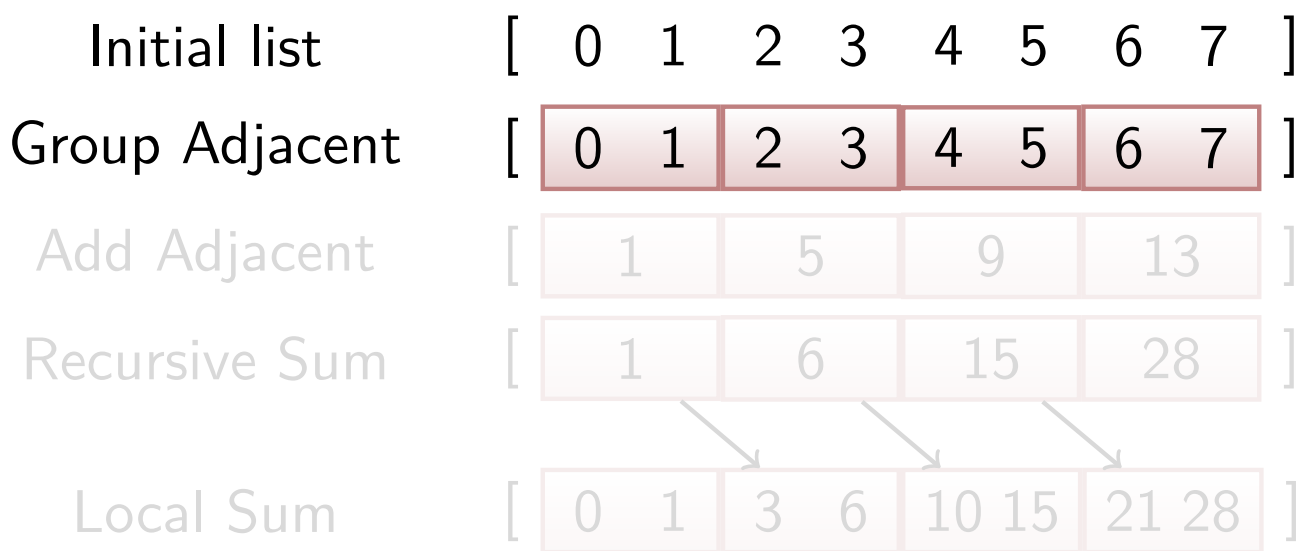

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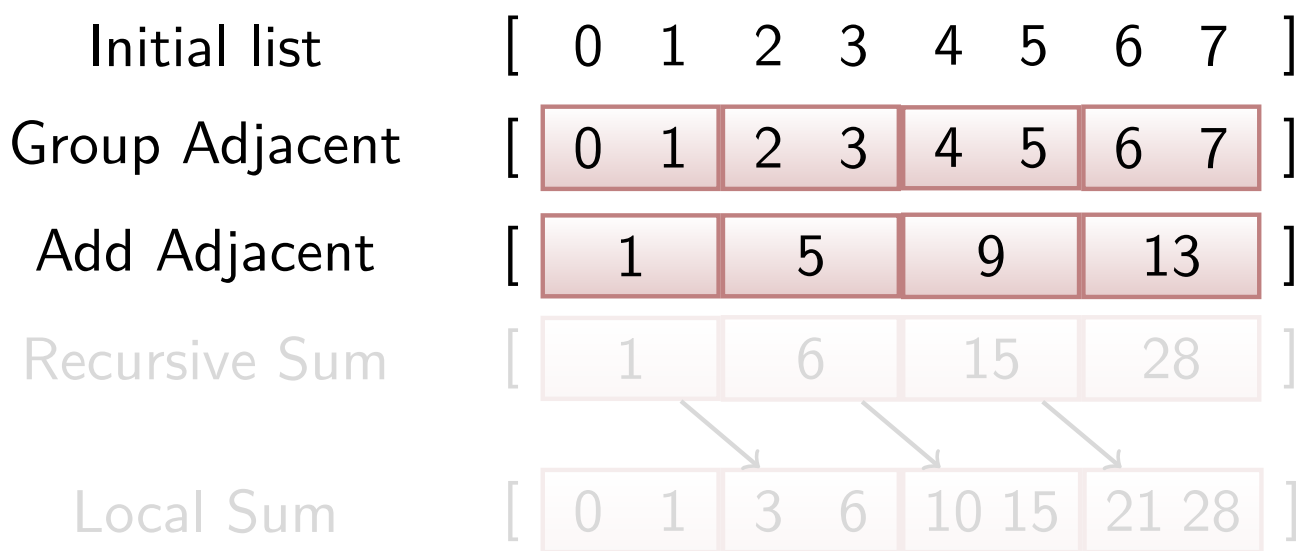

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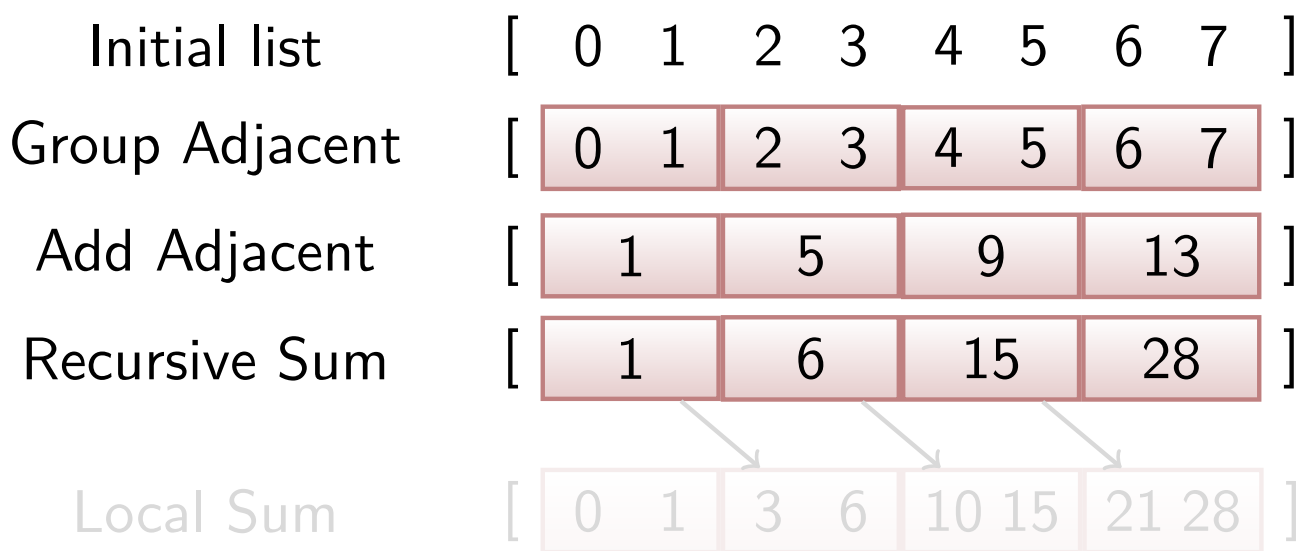

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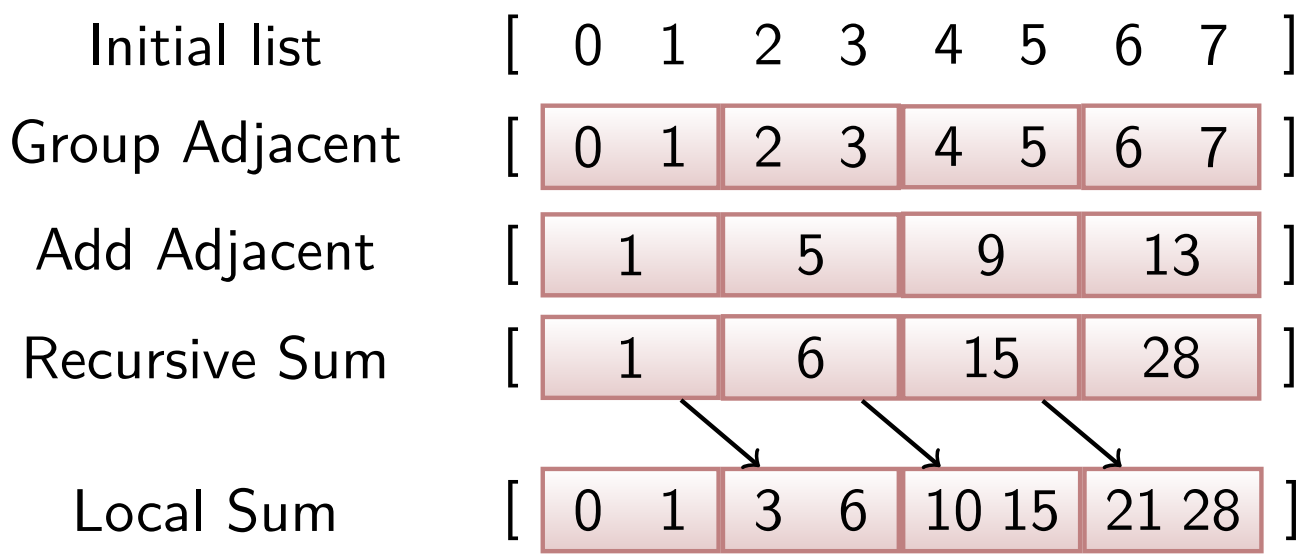

Ladner and Fischer Prefix Sums

```
op lf : Pow -> Pow .
eq lf([ N ]) = [ N ] .
eq lf(P zip Q) = (rsh(0, lf(P ++ Q)) ++ P) zip lf(P ++ Q) .

--- Right shift.
op rsh : Nat Pow -> Pow .
eq rsh(M, [ N ]) = [ M ] .
eq rsh(M, P tie Q) = rsh(M, P) tie rsh(last(P), Q) .

--- Sum each element.
op _++_ : [Pow] [Pow] -> [Pow] .
eq [ M ] ++ [ N ] = [ M + N ] .
eq (P1 tie P2) ++ (Q1 tie Q2) = (P1 ++ Q1) tie (P2 ++ Q2) .
```

Coverset Induction

- As an example, we want to show this theorem.

```
(goal ps-lf : POWERLIST-PREFIX |-
  A{P:Pow}   ((ps(P)) = (lf(P))) .)
```

- Will need to use recursive calls in the definition to generate induction schemes.

```
op lf : Pow -> Pow .
eq lf([ N ]) = [ N ] .
eq lf(P zip Q) = (rsh(0, lf(P ++ Q)) ++ P) zip lf(P ++ Q) .
```

Generalization to MEL

- Coverset induction was originally developed for unsorted and total functions.
- Prefix sum is only defined on well-structured powerlists.
- Our idea is to iteratively refine a *pattern* against equations. unifying with that pattern.
- Instantiate variables using memberships associated with variable sort constraint.

Example Induction

- Suppose we want to show the following lemma:

```
(lem last-zip :
  A{P:Pow ; Q:Pow}
    ((sim?(P,Q)) = (true)
      => (last(P zip Q)) = (last(Q))) .)
```

- Can show using coverset induction on `P zip Q`

```
op _zip_ : [Pow] [Pow] -> [Pow] .
eq (P tie Q) zip (R tie S) = (P zip R) tie (Q zip S) .
eq [M] zip [N] = [M] tie [N] .
```

Example Induction

- We start with pattern `P zip Q` which unifies with the left-hand sides of both equations.

```
op _zip_ : [Pow] [Pow] -> [Pow] .
eq (P tie Q) zip (R tie S) = (P zip R) tie (Q zip S) .
eq [M] zip [N] = [M] tie [N] .
```

- We can then instantiate these variables with the constructor memberships to yield 4 patterns.

```
cmb P tie Q : Pow if sim?(P, Q) = true .
cmb [N] : Pow if N : Nat.
```

- For each pattern, we generate a separate case. Induction hypotheses are added when the specialized pattern matches a recursive call.

Induction Cases 1/2

```
A{M:Nat ; N:Nat}
  ((sim?([M], [N])) = (true)
    => (last([M] zip [N])) = (last([N])))

A{P1:Pow ; P2:Pow ; N:Nat}
  ((sim?(P1 tie P2, [N])) = (true)
    => (last((P1 tie P2) zip [N])) = (last([N])))

A{M:Nat ; Q1:Pow ; Q2:Pow}
  ((sim?([M], Q1 tie Q2)) = (true)
    => (last([M] zip (Q1 tie Q2))) = (last(Q1 tie Q2)))
```

Induction Cases 2/2

```
A{P1:Pow ; P2:Pow ; Q1:Pow ; Q2:Pow}
  ((sim?(P1 tie P2, Q1 tie Q2)) = (true)
   & (sim?(P1, P2)) = (true)
   & (sim?(Q1, Q2)) = (true)
   & ((sim?(P1, Q1)) = (true)
         => (last(P1 zip Q1)) = (last(Q1)))
   & ((sim?(P2, Q2)) = (true)
         => (last(P2 zip Q2)) = (last(Q2)))
   => (last((P1 tie P2) zip (Q1 tie Q2))) = (last(Q1 tie Q2)))
```

Extensions

- With coverset induction, one can perform induction by matching against any set of complete memberships.

```
cmb P zip Q : Pow if sim?(P, Q) = true .
cmb [N] : Pow if N : Nat.
```

- Can further instantiate variables to match additional left-hand sides and enable more simplification via rewriting.

```
sim?(P zip Q, R)
```

Prefix sum example

- For the theorem

$$(\forall\, x\, :\, \mathrm{Pow})\ \mathrm{lf}(x) = \mathrm{ps}(x),$$

coverset induction using $\mathrm{lf}(x)$ using zip generates two cases:

```
eq lf([ N ]) = [ N ] .
eq lf(P zip Q) = (rsh(0, lf(P ++ Q)) ++ P) zip lf(P ++ Q) .
```

- Base case:

$$(\forall x : \mathrm{Nat})\ \mathrm{lf}([x]) = \mathrm{ps}([x]).$$

- Induction case:

$$(\forall p, q : \mathrm{Pow})\ \mathrm{sim?}(p, q) \wedge$$
$$\mathrm{lf}(p +\!\!+ q) = \mathrm{ps}(p +\!\!+ q) \implies \mathrm{lf}(p\ \mathrm{zip}\ q) = \mathrm{ps}(p\ \mathrm{zip}\ q).$$

Maude ITP Proof for Fischer-Ladner closure

- Required 8 basic lemmas about `sim?`, `tie`, and `zip`.
- One "typing" lemma about each defined operation:
  - `last`, element-wise addition, `rsh`, `ps`, and `lf`.
- Requires 6 lemmas to simplify terms with different operations.

Typing Lemma for Element-wise Addition

```
(lem list-sum-pow :
  A{P:Pow ; Q:Pow ; R:Pow}
   ((sim?(P,Q)) = (true)
     => (P ++ Q) : Pow
      & (sim?(P ++ Q, R)) = (sim?(P, R))) .)
(cov-split* on (P ++ Q) split (sim?(P, R)) .)
```

Proof Script 1/2

```
(lem last-zip :
  A{P:Pow ; Q:Pow}
    ((sim?(P,Q)) = (true) => (last(P zip Q)) = (last(Q))) .)
(cov* on P zip Q .)

(lem last-elt-sum :
  A{N:Nat ; P:Pow} ((last(N + P)) = (N + last(P))) .)
(cov* on last(P) .)

(lem elt-sum-zip :
  A{M:Nat ; P:Pow ; Q:Pow}
   (  (sim?(P, Q)) = (true)
    => (M + (P zip Q)) = ((M + P) zip (M + Q))) .)
(cov* on P zip Q .)
```

Proof Script 2/2

```
(lem rsh-self-elt-plus-general :
  A{M:Nat ; N:Nat ; P:Pow}
   ((rsh(N + M, N + P)) = (N + rsh(M, P))) .)
(cov* on N + P .)

(lem rsh-self-elt-plus :
  A{N:Nat ; P:Pow}
   ((rsh(N, N + P)) = (N + rsh(0, P))) .)
(cov* on N + P .)

(lem ps-zip :
  A{P:Pow ; Q:Pow}
   ((sim?(P,Q)) = (true)
    => (ps(P zip Q))
         = ((rsh(0, ps(P ++ Q)) ++ P) zip ps(P ++ Q))) .)
(cov* on P ++ Q .)
```

Comparison with Related Work

- Misra's definition of prefix sum requires 1 journal page.
- Misra derives Ladner-Fischer from definition in 2.5 journal pages.
- Maude ITP Proof differs from Misra's, but uses same functions.
- Complete prefix sum spec is ~ 60 lines, and proof scripts are ~ 120 lines.
- ACL2 proof involves ~ 400 lines of proof, and several auxiliary definitions.

Summary

- We have made significant improvements to the Maude ITP including
  - Coverset induction
  - Equivalence propagation
  - Additional commands for developing a proof strategy and debugging proofs.
- We have validated these extensions with an extensive case study on formalizing parallel algorithms using powerlists.
  - Correctness of Fast Fourier transform, Batcher sort, prefix sums, and adder circuits.
  - Theorems require these new features in ITP.
  - Total case study: ~ 560 lines of specification and ~ 1320 lines of proof.

Thanks for listening