Trusted Extension in Coq

Laurent Théry

INRIA Sophia Antipolis France

Motivations

Why trust is so important?

Mainly Proving Truth

Keep the design simple

$\longrightarrow$ Proving Trusted Base

CCP (Code-Carrying Proofs)

Genericity $\rightarrow$ Domain Specific Applications

Outline

Overview of Coq

Reflection

Extensions: Various Flavours

More Extensions

Conclusions

Coq

Logic: Calculus of Inductive Constructions

First version 86, now `v8.2`

`http://coq.inria.fr/`

Trust in Coq

- Explicit Proof Object
- Trusted Component: `coqchk`

Coq

```
Inductive ℕ : Type := O | S (n:ℕ).

Inductive _ = _ : ∀T: Type, T → T → Prop :=
    refl_equal : ∀T: Type, ∀x: T, x = x.

Fact triv : 2 = 2.
Proof. apply refl_equal. Qed.

Print triv.
  triv = (refl_equal ℕ 2)
```

Coq

```
Function x + y :=
  if x is S x' then S(x' + y) else y.

Fact triv' : 1 + 2 = 3.
Proof. apply refl_equal. Qed.

Print triv'.

  triv' = (refl_equal ℕ 3)
```

Reflection

Samuel Boutin ('97)
"Using reflection to build efficient and certified decision procedures"

$$
\begin{aligned}
&(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)\,(m^2+n^2+o^2+p^2+q^2+r^2+s^2+t^2) = \\
&\qquad (am-bn-co-dp-eq-fr-gs-ht)^2+ \\
&\qquad (bm+an+do-cp+fq-er-hs+gt)^2+ \\
&\qquad (cm-dn+ao+bp+gq+hr-es-ft)^2+ \\
&\qquad (dm+cn-bo+ap+hq-gr+fs-et)^2+ \\
&\qquad (em-fn-go-hp+aq+br+cs+dt)^2+ \\
&\qquad (fm+en-ho+gp-bq+ar-ds+ct)^2+ \\
&\qquad (gm+hn+eo-fp-cq+dr+as-bt)^2+ \\
&\qquad (hm-gn+fo+ep-dq-cr+bs+at)^2
\end{aligned}
$$

Reflection

Abelian Group:

```
Inductive expr: Type :=
 add (e₁ e₂: expr) | opp (e: expr) | var (n: ℕ) | zero.
```

$(x + y) - y \rightsquigarrow$ `(add (add (var 1) (var 2)) (opp (var 1)))`

List of integers

```
Function l₁ ++ l₂ :=
 if l₁ is a::l₃ then
 if l₂ is b::l₄ then
           if |a| < |b| then a::(l₃ ++ l₂)
   else if |b| < |a| then b::(l₁ ++ l₄)
   else if    a == b  then a::b::(l₃ ++ l₄)
   else (l₃ ++ l₄)
 else l₁
 else l₂
```

`[1,1,2] ++ [-1,2]` $\rightsquigarrow$ `[1,2,2]`

Reflection

Normalisation:

```
Function e2l e :=
 match e with
 | add e1 e2 ⇒ e2l e1 ++ e2l e2
 | opp e1    ⇒ map (fun x ⇒ −x) (e2l e1)
 | var n     ⇒ [+n]
 | zero      ⇒ []
 end
```

e2l (add (add (var 1) (var 2)) (opp (var 2))) $\rightsquigarrow$

([1] ++ [2]) ++ (map (fun $x \Rightarrow -x$) [2])

[1,2] ++ (map (fun $x \Rightarrow -x$) [2])

[1,2] ++ [-2]

[1]

Reflection

Interpretations:

```
Structure Group :=
    {
        T:   Type;
      add:   T -> T -> T;
      opp:   T -> T;
     zero:   T;
    assoc:   ∀x∀y∀z, add (add x y) = add x (add y z);
      com:   ∀x∀y, add x y = add y x;
    lefti:   ∀x, add x zero = x;
    lefto:   ∀x, add x (opp x) = zero;
    }
```

Definition zGroup := mkGroup $\{\mathbb{Z},+,-,0,\ldots\}$

Definition oGroup := mkGroup {bool, $\oplus$, $\neg$, true, ...}

Reflection

Interpretations:

```
Function [|e|]_{g,env} :=
 match e with
 | add e1 e2 ⇒ g.add [|e1|]_{g,env} [|e2|]_{g,env}
 | opp e1    ⇒ g.opp [|e1|]_{g,env}
 | var n     ⇒ env n
 | zero      ⇒ g.zero
 end
```

$[|\texttt{add (add (var 1) (var 2)) (opp (var 2))}|]_{\texttt{zGroup},\{1 \mapsto x, 2 \mapsto y\}}$

$$\rightsquigarrow (x + y) + -y$$

$[|\texttt{add (add (var 1) (var 2)) (opp (var 2))}|]_{\texttt{oGroup},\{1 \mapsto x, 2 \mapsto y\}}$

$$\rightsquigarrow (x \oplus y) \oplus \neg y$$

Reflection

Interpretations:

Definition $\texttt{<|}z\texttt{|>}_{g,env}$:=
 if 0 < z then env z else g.opp (env $-z$)

Function $\texttt{\{|}l\texttt{|\}}_{g,env}$:=
 match l with
 | [] $\Rightarrow$ g.zero
 | [z] $\Rightarrow$ $\texttt{<|}z\texttt{|>}_{g,env}$
 | z::l_1 $\Rightarrow$ g.add $\texttt{<|}z\texttt{|>}_{g,env}$ $\texttt{\{|}l_1\texttt{|\}}_{g,env}$
 end

$\texttt{\{|[1]|\}}_{\texttt{zGroup},\{1\mapsto x,2\mapsto y\}} \rightsquigarrow x$

$\texttt{\{|[1]|\}}_{\texttt{oGroup},\{1\mapsto x,2\mapsto y\}} \rightsquigarrow x$

Reflection

Correctness:

Lemma e2l_cor: $\forall g \, \forall env \, \forall e$, $[|e|]_{g,env}$ = $\{|\texttt{e2l}\ e|\}_{g,env}$.

Proof by structural induction on e

Proof of $\forall x \forall y,\ (x+y)+-y=x$:

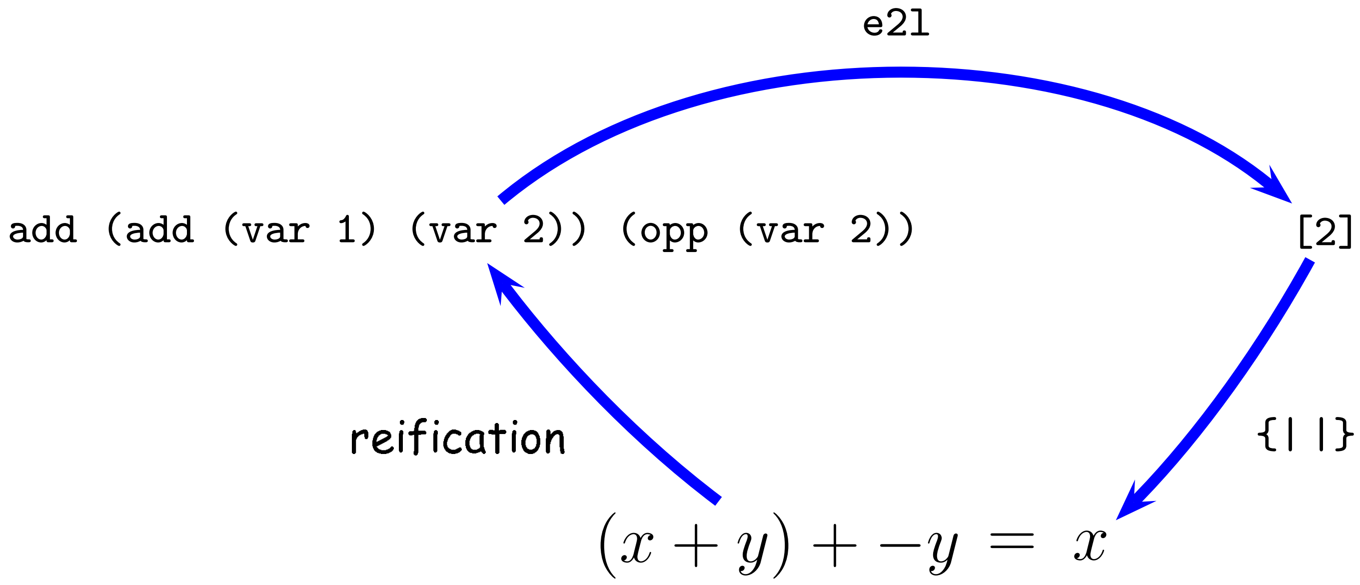
```
(fun x y ⇒
   (e2l_cor zGroup {1↦x,2↦y}
   (add (add (var 1) (var 2)) (opp (var 2)))))
```

Proof of $\forall x \forall y,\ (x \oplus y) \oplus \neg y = x$:

```
(fun x y ⇒
   (e2l_cor oGroup {1↦x,2↦y}
   (add (add (var 1) (var 2)) (opp (var 2)))))
```

Reflection

Summary:



Reflection

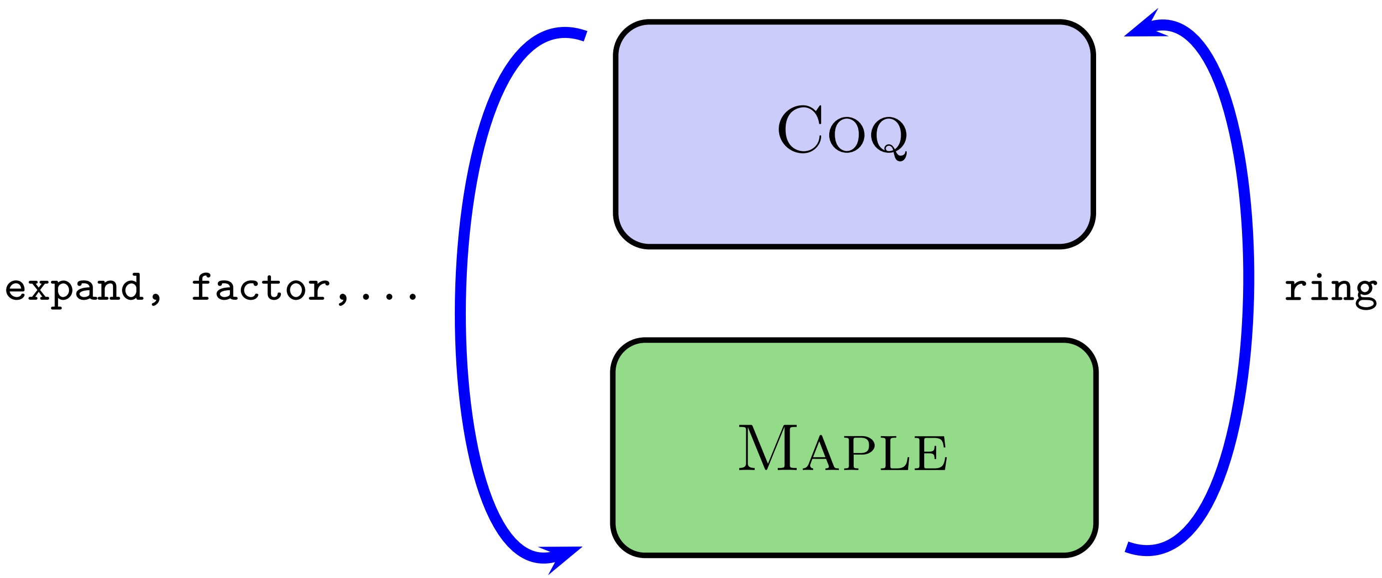
Degen's identity: ('97) 5m, (`v8.2`) 0,1 s

Benjamin Grégoire, Assia Mahboubi ('05)
"Proving equalities in a commutative ring done right in Coq"

Benjamin Grégoire, Xavier Leroy ('02)
"A compiled implementation of strong reduction"

Extension

David Delahaye, Micaela Mayero ('05)
" Dealing with algebraic expressions over a field in Coq using Maple"



Extension

Laurent Théry ('05)

"Simplifying Polynomial Expressions in a Proof Assistant"

$$y(\cancel{3x} + y + 2) < x(\cancel{3y} + 1) + z(x + y)$$

Normalisation:

$$0 < x - yy - 2y + zx + zy$$

Interactive:

$$y(y + 2) < x + z(x + y)$$

Extension

Frédéric Besson ('07)

"Fast Reflexive Arithmetics Tactics the linear case and beyond"

$$E = c \mid cx \mid E_1 + E_2$$

$$E_1 \leq E_2 \quad E_1 < E_2 \quad E_1 = E_2$$

$$\begin{cases} -2x & - & y & \geq & -1 \\ x & + & y & \geq & 2 \\ & - & y & \geq & -1 \end{cases}$$

Extension

Farka's Lemma:

Exactly one of the following statements holds:

$\exists x, A\,.\,x \geq b$

$\exists y, y \geq 0 \wedge A^y\,.\,y = 0 \wedge b^t\,.\,y > 0$

$$\left\{\begin{array}{rcrcrcl} -2x & - & y & \geq & -1 & \times & 1 \\ x & + & y & \geq & 2 & \times & 2 \\ & - & y & \geq & -1 & \times & 1 \end{array}\right.$$

$$0 \geq 2$$

Extension

John Harrison ('07)

"Verifying nonlinear real formulas via sums of square"

$$\forall x_1, \ldots, x_n,$$
$$P_1(x_1, \ldots, x_n) \geq 0 \quad \wedge \; \ldots \; \wedge \quad P_k(x_1, \ldots, x_n) \geq 0 \quad \Rightarrow$$
$$P(x_1, \ldots, x_n) \geq 0$$

Extension

Basic Idea:

$$P(x_1, \ldots, x_n)^2 \;\geq\; 0$$

$$P(x_1, \ldots, x_n) \;\geq\; 0 \quad \wedge \quad P(x_1, \ldots, x_n) \;\geq\; 0 \quad \Rightarrow$$
$$P(x_1, \ldots, x_n) + Q(x_1, \ldots, x_n) \;\geq\; 0$$

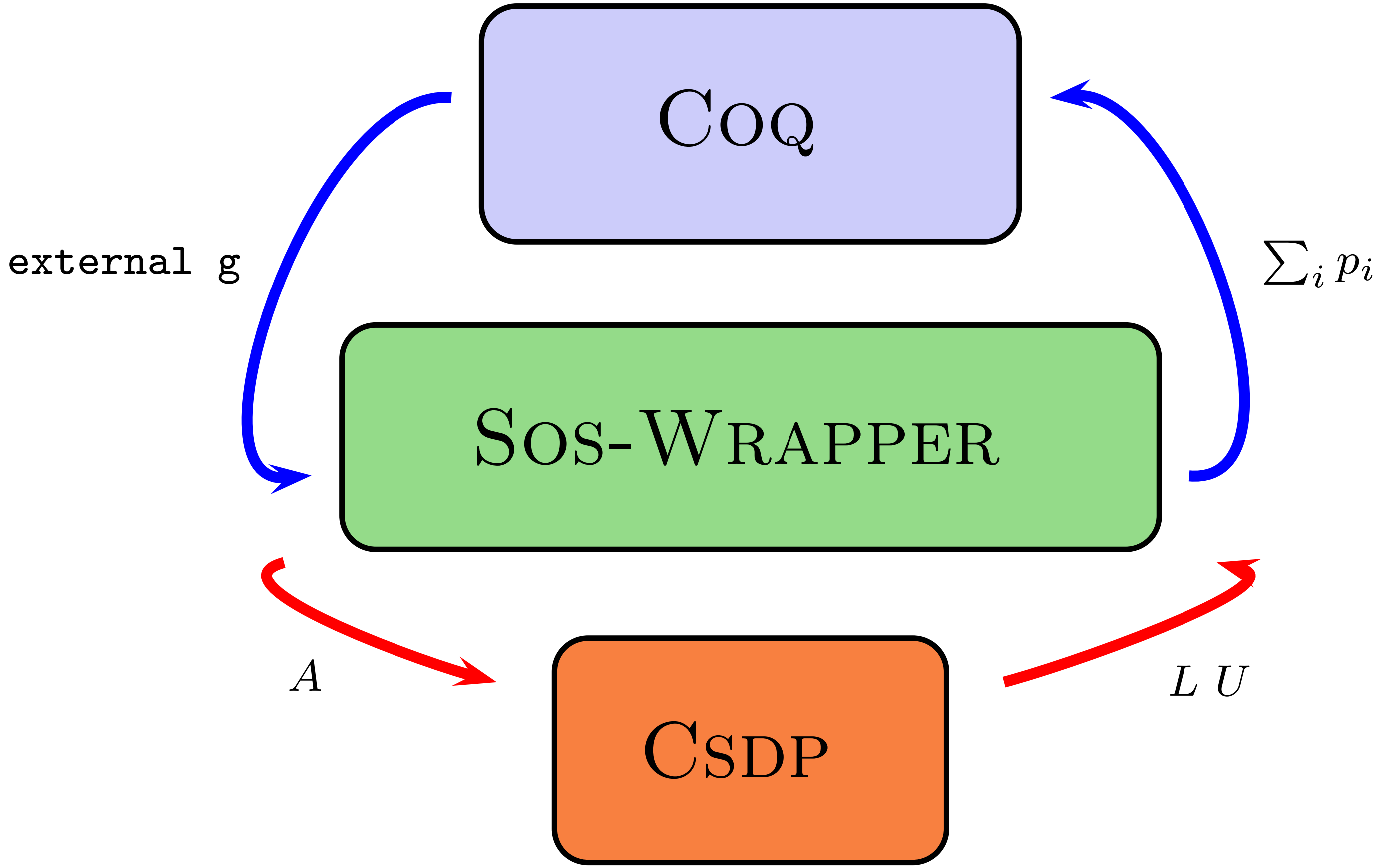
$$P(x_1, \ldots, x_n) \;\geq\; 0 \quad \wedge \quad Q(x_1, \ldots, x_n) \geq 0 \quad \Rightarrow$$
$$P(x_1, \ldots, x_n)\; Q(x_1, \ldots, x_n) \;\geq\; 0$$

Example

$$\forall a\, b\, c\, x, \quad ax^2 + bx + c = 0 \quad \Rightarrow \quad b^2 - 4ac \;\geq\; 0$$

$$b^2 - 4ac = (2ax + b)^2 - 4a(ax^2 + bx + c)$$

Extension



Extension

Robert S. Boyer, J Strother Moore ('81)

"The Mechanical Verification of a Fortran Square Root Program"

```
    INTEGER FUNCTION ISQRT(I)
    INTEGER I
    IF ((I .LT. 0)) STOP
    IF ((I .GT. 1)) GOTO 100
    ISQRT = I
    RETURN
100 ISQRT = (I / 2)
200 CONTINUE
    IF (((I / ISQRT) .GE. ISQRT)) RETURN
    ISQRT = ((ISQRT + (I / ISQRT)) / 2)
    GOTO 200
    END
```

Extension

Main theorem:

$$\forall i\, j,\ 0 \ \le\ i\ \wedge\ 0\ <\ j\ \Rightarrow\ i\ <\ ((j + i/j)/2 + 1)^2$$

Reduction: $k = i/j$ and $l = (j + k)/2$

$$\forall j\, k\, l,$$
$$0\ \le\ j\ \wedge 0\ \le\ k\ \wedge 0\ \le\ l\ \wedge 2l\ \le\ j{+}k\ \wedge j + k\ <\ 2(l + 1)$$
$$\Rightarrow\ jk + j\ \le\ (l + 1)^2$$

Sum of Squares:

$$4((l + 1)^2 - (jk + j)) =$$
$$(k - j + 1)^2 + (2(l + 1) - (j + k + 1))(3 + j + k + 2l))$$

Extension

Benjamin Grégoire, Loïc Pottier, Laurent Théry ('09)
"Proofs Certificates for Algebra and their Application to Automatic Geometry Theorem Proving"

$$\forall x_1, \ldots, x_n, \quad P_1(x_1, \ldots, x_n) = 0 \quad \wedge \ldots \wedge \quad P_k(x_1, \ldots, x_n) = 0 \quad \Rightarrow \quad P(x_1, \ldots, x_n) = 0$$

Hilbert's Nullstellensatz

$$cP(x_1, \ldots, x_n)^r = \sum_i Q_i(x_1, \ldots, x_n) P_i(x_1, \ldots, x_n)$$

Extension

Basic Idea

$$P(x_1, \ldots, x_n)^r \in \{P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n)\}$$

Example

$$x^2y^2 - y^4 \in \{x^2 + 1, xy - 1\}$$

Division

$$x^2y^2 - y^4 = y^2(x^2 + 1) - y^4 - y^2$$

$$x^2y^2 - y^4 = (xy + 1)(xy - 1) - y^4 + 1$$

Spolynomial

$$x^2y = y(x^2 + 1) = x(xy - 1)$$

$$spoly(x^2 + 1, xy + 1) = y(x^2 + 1) - x(xy - 1) = x + y$$

Extension

Gröbner basis

$$\{x^2 + 1, xy - 1\}$$

$$spoly(x^2 + 1, xy + 1) = x + y$$

$$\{x^2 + 1, xy - 1, x + y\}$$

$$spoly(x^2 + 1, x + y) = x^2 + 1 - x(x + y) = -(xy - 1)$$
$$spoly(xy - 1, x + y) = xy - 1 - y(x + y) = -y^2 - 1$$

$$\{x^2 + 1, xy - 1, x + y, -y^2 - 1\}$$

$$spoly(xy - 1, -y^2 - 1) = y(xy - 1) - (-x)(-y^2 - 1) = -(x + y)$$

Extension

Certificate

$$x^2y^2 - y^4 \in \{x^2+1, xy-1\}$$
$$x^2y^2 - y^4 \in \{x^2+1, xy-1, x+y, -y^2-1\}$$

$$x^2y^2 - y^4 = y^2(x^2+1) - y^4 - y^2$$
$$-y^4 - y^2 = y^2(-y^2-1) + 0$$

$$x^2y^2 - y^4 = y^2(x^2+1) + y^2(-y^2-1)$$
$$x^2y^2 - y^4 = y^2(x^2+1) + y^2(xy-1-y(x+y))$$
$$x^2y^2 - y^4 = y^2(x^2+1) + y^2(xy-1-y(y(x^2+1)-x(xy-1)))$$
$$x^2y^2 - y^4 = (-y^4+y^2)(x^2+1) + (xy^3+y^2)(xy-1)$$

Extension

Straight-Line Program

$$X^{f_{n+1}} - 1 = 0 \wedge X^{f_n} - 1 = 0 \Rightarrow X - 1 = 0$$

where f_n: $f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n$

$$X - 1 = P_{n-2}(X^{f_{n+1}} - 1) + P_{n-1}(X^{f_n} - 1)$$

where P_n: $P_0 = 0, P_1 = 1, P_n = -X^{f_n} P_{n-1} + P_{n-2}$

$$P_1 = X^{f_{n+1}} - 1 \quad P_2 = X^{f_n} - 1$$

$$C = [[1, -X^{f_{n-1}}] \qquad P_3 = P_1 - X^{f_{n-1}} P_2$$

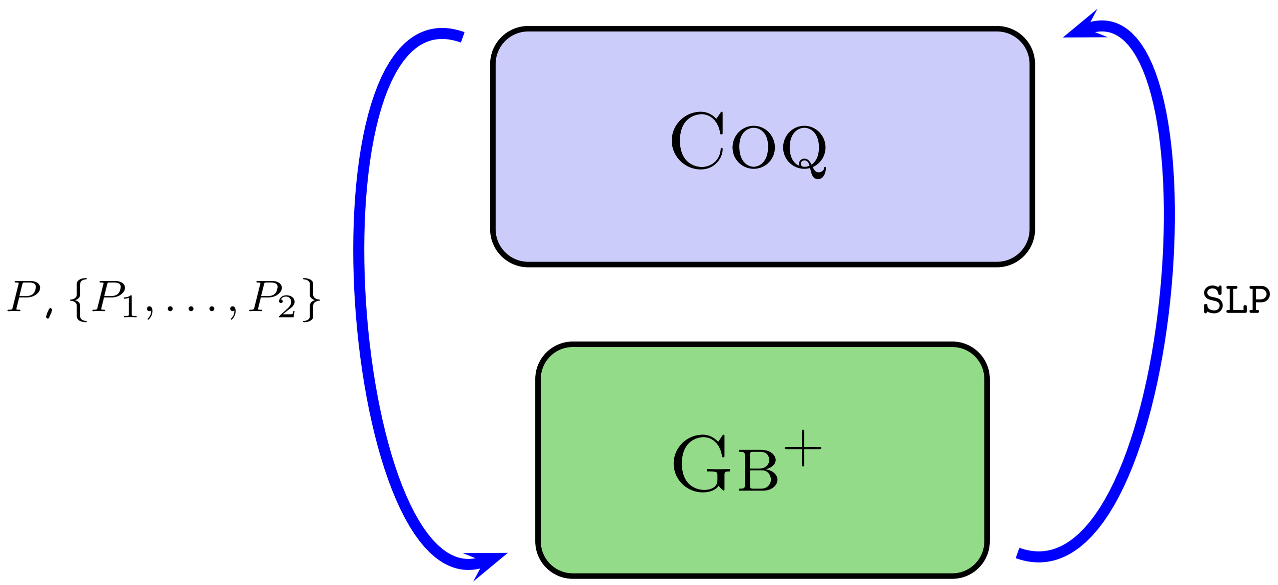
$$[0, 1, -X^{f_{n-2}}], \qquad P_4 = P_2 - X^{f_{n-2}} P_3$$

$$\ldots \qquad \ldots$$

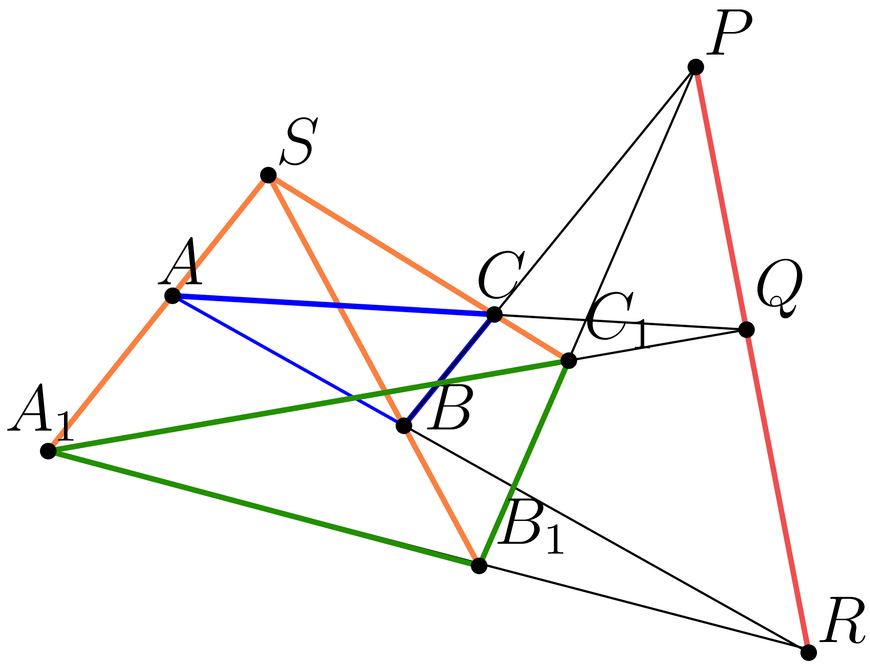
$$[0, \ldots, 0, 1, -X^{f_2}]] \qquad P_n = P_{n-2} - X^{f_2} P_{n-1}$$

$$CR = [0, \ldots, 0, 1] \qquad X - 1 = P_n$$

Extension



Extension




```
Record point := {x: ℝ, y: ℝ}.
Definition collinear (A B C:point):=
  (A.x - B.x) * (C.y - B.y) - (A.y - B.y) * (C.x - B.x) = 0.
Definition parallel (A B C D:point):=
  (A.x - B.x) * (C.y - D.y) = (A.y - B.y) * (C.x - D.x).
```

Extension

	Time (seconds)		Size (characters)		Size (nodes)
Theorem	Computing	Verifying	Polynomials	Certificate	SLP
Ceva	181	2.5	538644	477414	76669
Desargues	0.3	0.01	6359	4551	4311
Feuerbach	0.8	0.4	52569	16999	5497
Pappus	1.3	0.2	2721	1934	8031
Pascal	397	12	732982	864509	183505
Ptolemy	200	2.4	571931	571931	73257
Simson	0.3	0.2	1541	1238	4919
Thales	0.03	0.1	5422	5169	1323

More Extensions

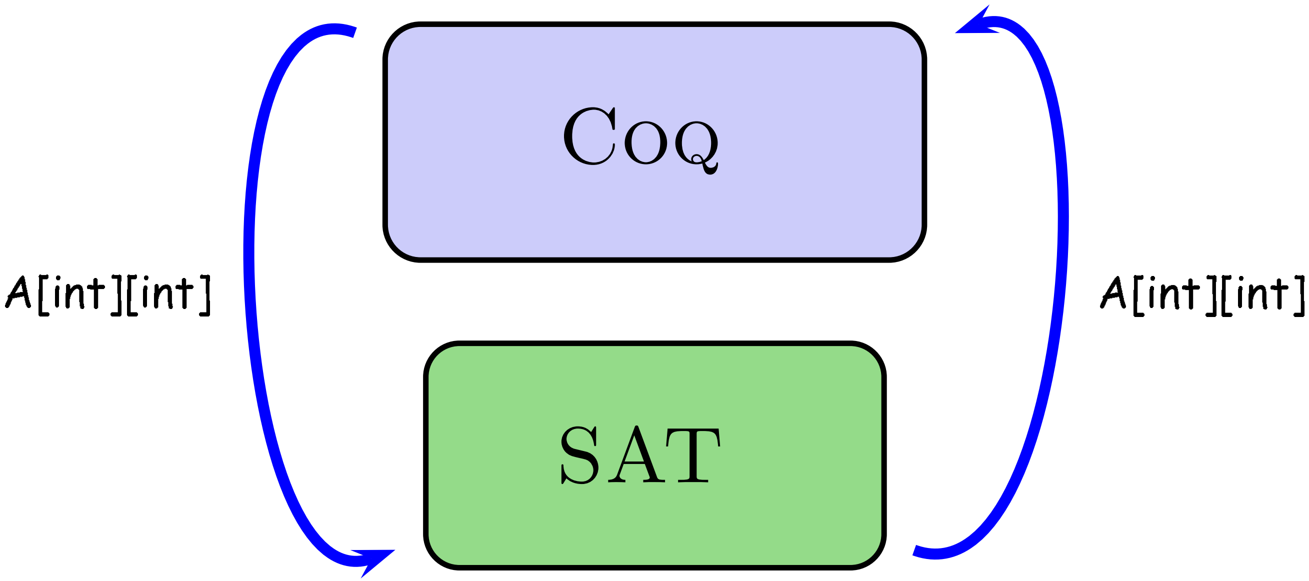
Michaël Armand, Benjamin Grégoire, Arnaud Spiwack and Laurent Théry ('10)

"Extending Coq with Imperative Features and its Application to SAT Verification"

Chantal Keller and Benjamin Werner ('10)

"Importing HOL Light into Coq"

Extension



More Extensions

```
p cnf 50 80
 16   23   42 0
-16   23   42 0
 26   41  -42 0
-26   41  -42 0
 32  -41  -42 0
  6   15  -41 0
 -6   15  -32 0
  1  -32   46 0
 -1  -32   46 0
-15  -41  -46 0
-15  -21  -46 0
-23   33   38 0
-23  -33   38 0
  8   22   33 0
  8   22  -33 0
-22   37  -38 0
 13   36  -37 0
 13  -22  -36 0
-13  -22  -37 0
 11  -23   47 0
 -8   11  -47 0
 -8  -11   39 0
-11   27  -39 0
 -8  -11  -39 0
 -7   26   29 0
 -7  -26   29 0
-13   20   36 0
-13   17   20 0
  5  -17   20 0
  5  -19  -45 0
 -5  -10  -45 0
  6   25   47 0
 -6  -10   25 0
 -2  -27   37 0
-27  -36   40 0
 18   39  -40 0
 -2  -19   31 0
  5   18  -30 0
-31  -43  -50 0
 10  -30   43 0
 10  -41   43 0
 19   21   29 0
 37   42   45 0
-20   27   40 0
-21  -36   48 0
 31  -36  -48 0
  3   -9  -18 0
 16  -40  -47 0
  1  -18   21 0
  2   28   32 0
 -1  -24  -50 0
-12   35   49 0
 -6  -36   45 0
  7   12  -43 0
  7   30  -43 0
 -5    9  -17 0
  3   14   50 0
-12   17  -49 0
 24   34   49 0
 14  -20   24 0
 -9   35  -49 0
 -4  -47   50 0
  4   44  -44 0
 28  -28  -38 0
  2    4  -48 0
-20   35  -44 0
 30  -31  -43 0
-14  -29   35 0
-20   35  -35 0
 19  -22  -24 0
 25  -28   48 0
-14  -34   44 0
  9   20   44 0
 -3    9  -29 0
 17   34  -34 0
 12   48   48 0
-12  -25  -43 0
-25  -31   48 0
 14  -16   49 0
 -3   -4  -35
```

```
p cnf 50 80
 16   23   42 0
-16   23   42 0
 26   41  -42 0
-26   41  -42 0
```

More Extensions

Resolution: $\dfrac{x \vee C \qquad \neg x \vee C'}{C \vee C'}$

```
 3  2
 1  0
14 13
21 23
20 19 79
12 11
18 16 15 81 78 80
 8  7
 5  6  9 83  4 76
77 84
80 85
17 88 89
15 87 89
18 91 89 90
```

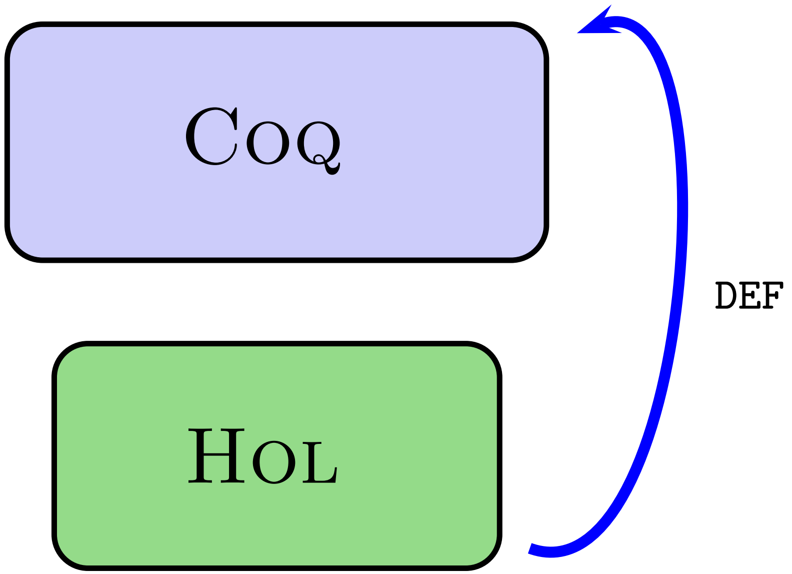
More Extensions

Problem	Vars	Clauses	zChaff	zVerify
dubois50	150	400	0.00	0.01
barrel5	1407	5383	0.50	0.07
barrel6	2306	8931	1.74	0.14
barrel7	3523	13765	5.20	0.26
6pipe	15800	394739	42.21	2.86
longmult14	7176	22390	408.55	7.34
hole11	132	738	14.82	0.90
hole12	156	949	144.49	4.85
hole13	182	1197	5048.23	-

More Extensions

Problem	zVerify	Coq	Cert	Typing	Check
dubois50	0.01	0.04	0.00	0.02	0.02
barrel5	0.07	0.47	0.00	0.32	0.15
barrel6	0.14	1.15	0.08	0.62	0.45
barrel7	0.26	1.45	0.17	0.80	0.48
6pipe	2.86	24.73	0.98	13.92	9.83
longmult14	7.34	73.63	7.72	27.07	38.84
hole11	0.90	0.41	2.96	6.14	1.39
hole12	4.85	58.28	2.44	18.47	37.38
hole13	-	1068.30	88.15	387.44	592.72

More Extensions



More Extensions

	Number		Time			Memory	
Bench.	Theorems	Lemmas	Rec.	Exp.	Comp.	H.D.D.	Virt. Coq
Stdlib	1,726	195,317	2 min 30	6 min 30	10h	218 Mb	4.5 Gb
Model	2,121	322,428	6 min 30	29 min	44h	372 Mb	7.6 Gb
Vectors	2,606	338,087	6 min 30	21 min	39h	329 Mb	7.5 Gb

Conclusions

Key Role of Certificate

Granularity

Clear Separation

Room for Improvement