Appendix to “Iterated Ultrapowers for the Masses”

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August 18, 2017

Abstract

This document provides solutions to the exercises in “Iterated Ultrapowers for the Masses”.

3.3. Exercise. Let \( \mathcal{U} \) be an \( \mathcal{M} \)-amenable ultrafilter on the parametrically \( \mathcal{M} \)-definable subsets of \( \mathcal{M} \). We begin by showing that \( \mathcal{U}^2 \) is a filter on the parametrically \( \mathcal{M} \)-definable subsets of \( \mathcal{M}^2 \). Let \( X, Y \subseteq \mathcal{M}^2 \) be parametrically \( \mathcal{M} \)-definable. Let

\[
Z = \{ m \in \mathcal{M} : (X \cap Y) | m \in \mathcal{U} \},
\]

\[
Z_1 = \{ m \in \mathcal{M} : X | m \in \mathcal{U} \},
\]

\[
Z_2 = \{ m \in \mathcal{M} : Y | m \in \mathcal{U} \}.
\]

Since \( \mathcal{U} \) is \( \mathcal{M} \)-amenable, \( Z, Z_1 \) and \( Z_2 \) are parametrically \( \mathcal{M} \)-definable subsets of \( \mathcal{M} \). Now,

\[
Z = \{ m \in \mathcal{M} : X | m \cap Y | m \in \mathcal{U} \}.
\]

And, since \( \mathcal{U} \) is an ultrafilter, for all \( m \in \mathcal{M} \),

\[
X | m \cap Y | m \in \mathcal{U} \text{ if and only if } X | m \notin \mathcal{U} \text{ and } Y | m \in \mathcal{U}.
\]

Therefore, using the fact that \( \mathcal{U} \) is a filter,

\[
X \cap Y \in \mathcal{U}^2 \iff Z \in \mathcal{U} \iff Z_1 \cap Z_2 \in \mathcal{U} \iff Z_1 \in \mathcal{U} \text{ and } Z_2 \in \mathcal{U} \iff X \in \mathcal{U}^2 \text{ and } Y \in \mathcal{U}^2.
\]

It follows that if \( X, Y \in \mathcal{U}^2 \), then \( X \cap Y \in \mathcal{U}^2 \); and if \( X \in \mathcal{U}^2 \) and \( X \subseteq Y \), then \( Y \in \mathcal{U}^2 \). Therefore \( \mathcal{U}^2 \) is a filter. We are left to verify that \( \mathcal{U}^2 \) is an ultrafilter. Let \( X \) and \( Z_1 \) be as above. Let

\[
W = \{ m \in \mathcal{M} : (M^2 \setminus X) | m \in \mathcal{U} \}.
\]

Again, since \( \mathcal{U} \) is \( \mathcal{M} \)-amenable, \( W \) is a parametrically \( \mathcal{M} \)-definable subset of \( \mathcal{M} \). Note that

\[
W = \{ m \in \mathcal{M} : M \setminus (X | m) \in \mathcal{U} \}.
\]

And, since \( \mathcal{U} \) is an ultrafilter, for all \( m \in \mathcal{M} \),

\[
M \setminus (X | m) \in \mathcal{U} \text{ if and only if } X | m \notin \mathcal{U}.
\]

Therefore

\[
W = \{ m \in \mathcal{M} : X | m \notin \mathcal{U} \} = M \setminus Z_1.
\]
And so, since $\mathcal{U}$ is an ultrafilter,

$$M^2 \setminus X \in \mathcal{U} \iff W \in \mathcal{U} \iff Z_1 \notin \mathcal{U} \iff X \notin \mathcal{U}^2.$$ 

This shows that $\mathcal{U}^2$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^2$.

**3.7. Exercise.** Let $\mathcal{U}$ be an $\mathcal{M}$-amenable ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M$. We will prove that for all positive integers $n$, $\mathcal{U}^n$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^n$ by induction on $n$. The proof is an obvious generalization of Exercise 3.3. Let $n > 0$ and suppose that $\mathcal{U}^n$ is an ultrafilter on the parametrically $M^n$-definable subsets of $M^n$. We need to show that $\mathcal{U}^{n+1}$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^{n+1}$. We begin by showing that $\mathcal{U}^{n+1}$ is a filter. Let $X, Y \subseteq M^{n+1}$ be parametrically $\mathcal{M}$-definable. We will prove that for all positive integers $n$, $\mathcal{U}^n$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^n$ by induction on $n$. The proof is an obvious generalization of Exercise 3.3. Let $n > 0$ and suppose that $\mathcal{U}^n$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^n$. We need to show that $\mathcal{U}^{n+1}$ is a filter. Let $X, Y \subseteq M^{n+1}$ be parametrically $\mathcal{M}$-definable. Let

$$Z = \{ m \in M : (X \cap Y)|m \in \mathcal{U}^n \},$$

$$Z_1 = \{ m \in M : X|m \in \mathcal{U}^n \},$$

$$Z_2 = \{ m \in M : Y|m \in \mathcal{U}^n \}.$$ 

It follows from Lemma 3.6 that $Z$, $Z_1$ and $Z_2$ are parametrically $\mathcal{M}$-definable subsets of $M$. Now,

$$Z = \{ m \in M : X|m \cap Y|m \in \mathcal{U}^n \}.$$ 

By the induction hypothesis, for all $m \in M$,

$$X|m \cap Y|m \in \mathcal{U}^n \text{ if and only if } X|m \in \mathcal{U}^n \text{ and } Y|m \in \mathcal{U}^n.$$ 

Therefore, using the fact that $\mathcal{U}$ is a filter,

$$X \cap Y \in \mathcal{U}^{n+1} \iff Z \in \mathcal{U} \iff Z_1 \cap Z_2 \in \mathcal{U} \iff$$

$$Z_1 \in \mathcal{U} \text{ and } Z_2 \in \mathcal{U} \iff X \in \mathcal{U}^{n+1} \text{ and } Y \in \mathcal{U}^{n+1}.$$ 

It follows that if $X, Y \in \mathcal{U}^{n+1}$, then $X \cap Y \in \mathcal{U}^{n+1}$; and if $X \in \mathcal{U}^{n+1}$ and $X \subseteq Y$, then $Y \in \mathcal{U}^{n+1}$. This shows that $\mathcal{U}^{n+1}$ is a filter. We are left to verify that $\mathcal{U}^{n+1}$ is an ultrafilter. Let $X$ and $Z_1$ be as above. Let

$$W = \{ m \in M : (M^{n+1} \setminus X)|m \in \mathcal{U}^{n+1} \}.$$ 

It follows from Lemma 3.6 that $W$ is a parametrically $\mathcal{M}$-definable subset of $M$. Note that

$$W = \{ m \in M : M^n \setminus (X|m) \in \mathcal{U}^n \}.$$ 

By the induction hypothesis we have, for all $m \in M$,

$$M^n \setminus (X|m) \in \mathcal{U}^n \text{ if and only if } X|m \notin \mathcal{U}^n.$$ 

Therefore

$$W = \{ m \in M : (X|m) \notin \mathcal{U}^n \} = M \setminus Z_1.$$ 

Using the fact that $\mathcal{U}$ is an ultrafilter, this shows that

$$M^{n+1} \setminus X \in \mathcal{U}^{n+1} \iff W \in \mathcal{U} \iff Z_1 \notin \mathcal{U} \iff X \notin \mathcal{U}^{n+1},$$ 

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which proves that $\mathcal{U}^{n+1}$ is an ultrafilter on the parametrically $\mathcal{M}$-definable subsets of $M^{n+1}$.

3.8. Exercise. (a). Let $\mathcal{U}$ be an $\mathcal{M}$-amenable ultrafilter. Let $\mathcal{M}^* = \text{Ult}(\mathcal{M}, \mathcal{U}, 2)$ (defined in Subsection ??). We need to define the interpretations of the relation symbols in $\mathcal{L}(\mathcal{M})$ in $\mathcal{M}^*$. Let $R(x_1, \ldots, x_n)$ be an $n$-ary relation symbol in $\mathcal{L}(\mathcal{M})$. For all $[f_1(0, 1)], \ldots, [f_n(0, 1)] \in \mathcal{M}^*$, define

$$R^{\mathcal{M}^*}([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ if and only if}$$

$$\{ (x, y) \in M^2 : \mathcal{M} \models R(f_1(x, y), \ldots, f_n(x, y)) \} \in \mathcal{U}^2.$$  

To see that this definition is consistent, let $f_1(x, y), \ldots, f_n(x, y), g_1(x, y), \ldots, g_n(x, y)$ be functions such that for all $1 \leq i \leq n$, $[f_i(0, 1)] = [g_i(0, 1)]$. Now, for all $1 \leq i \leq n$,

$$Z_i = \{ (x, y) \in M^2 : \mathcal{M} \models f_i(x, y) = g_i(x, y) \} \in \mathcal{U}^2.$$  

Therefore $Z = Z_1 \cap \cdots \cap Z_n \in \mathcal{U}^2$ and

$$\begin{align*}
\{ (x, y) \in M^2 : \mathcal{M} \models R(f_1(x, y), \ldots, f_n(x, y)) \} & \cap Z \\
= \{ (x, y) \in M^2 : \mathcal{M} \models R(g_1(x, y), \ldots, g_n(x, y)) \} & \cap Z. 
\end{align*}$$

This shows that replacing the $f_i$s by the $g_i$s does not change the truth value of $R^{\mathcal{M}^*}([f_1(0, 1)], \ldots, [f_n(0, 1)])$.

Theorem 0.1 (Łoś Theorem) Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}(\mathcal{M})$-formula. For all $[f_1(0, 1)], \ldots, [f_n(0, 1)] \in \mathcal{M}^*$,

$$\mathcal{M}^* \models \phi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ if and only if}$$

$$\{ (x, y) \in M^2 : \mathcal{M} \models \phi(f_1(x, y), \ldots, f_n(x, y)) \} \in \mathcal{U}^2.$$  

Proof We prove this theorem by structural induction on $\phi$. Without loss of generality we may assume that $\phi$ only contains the logical connectives $\neg$ and $\wedge$, and the quantifier $\exists$. It follows from the definition of $\mathcal{M}^*$ in Subsection ?? and above that the theorem holds for all atomic formulae. Suppose that the theorem holds for $\psi(x_1, \ldots, x_n)$, and $\phi(x_1, \ldots, x_n) = \neg \psi(x_1, \ldots, x_n)$. Let $[f_1(0, 1)], \ldots, [f_n(0, 1)] \in \mathcal{M}^*$. Now,

$$\mathcal{M}^* \models \phi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ if and only if}$$

$$\neg (\mathcal{M}^* \models \psi([f_1(0, 1)], \ldots, [f_n(0, 1)])) \text{ if and only if}$$

$$\{ (x, y) \in M^2 : \mathcal{M} \models \psi(f_1(x, y), \ldots, f_n(x, y)) \} \notin \mathcal{U}^2 \text{ if and only if}$$

$$M^2 \setminus \{ (x, y) \in M^2 : \mathcal{M} \models \psi(f_1(x, y), \ldots, f_n(x, y)) \} \notin \mathcal{U}^2 \text{ if and only if}$$

$$\{ (x, y) \in M^2 : \mathcal{M} \models \neg \psi(f_1(x, y), \ldots, f_n(x, y)) \} \in \mathcal{U}^2.$$

Suppose that the theorem holds for $\psi(x_1, \ldots, x_n)$ and $\theta(x_1, \ldots, x_n)$, and $\phi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) \wedge \theta(x_1, \ldots, x_n)$. Let $[f_1(0, 1)], \ldots, [f_n(0, 1)] \in \mathcal{M}^*$. Let

$$Z_1 = \{ (x, y) \in M^2 : \mathcal{M} \models \psi(f_1(x, y), \ldots, f_n(x, y)) \},$$

$$Z_2 = \{ (x, y) \in M^2 : \mathcal{M} \models \theta(f_1(x, y), \ldots, f_n(x, y)) \},$$

$$Z = \{ (x, y) \in M^2 : \mathcal{M} \models \phi(f_1(x, y), \ldots, f_n(x, y)) \}.$$
Note that \( Z = Z_1 \cap Z_2 \). Now,

\[
\mathcal{M}^* \models \phi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ if and only if } \\
\mathcal{M}^* \models \psi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ and } \mathcal{M}^* \models \theta([f_1(0, 1)], \ldots, [f_n(0, 1)]) \text{ if and only if } \\
Z_1 \in U^2 \text{ and } Z_2 \in U^2 \text{ if and only if } Z \in U^2.
\]

Suppose that the theorem holds for \( \psi(w, x_1, \ldots, x_n) \), and \( \phi(x_1, \ldots, x_n) = \exists w \psi(w, x_1, \ldots, x_n) \).

Let \([f_1(0, 1)], \ldots, [f_n(0, 1)] \in \mathcal{M}^* \). Now, if \( \mathcal{M}^* \models \phi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \), then there exists \([h(0, 1)] \in \mathcal{M}^* \) such that

\[
\mathcal{M}^* \models \psi([h(0, 1)], [f_1(0, 1)], \ldots, [f_n(0, 1)]).
\]

Therefore

\[
\{(x, y) \in M^2 : \mathcal{M} \models \psi(h(x, y), f_1(x, y), \ldots, f_n(x, y)) \} \in U^2.
\]

And so

\[
\{(x, y) \in M^2 : \mathcal{M} \models \exists w \psi(w, f_1(x, y), \ldots, f_n(x, y)) \} \in U^2.
\]

We are left to show the converse. Since \( \mathcal{M} \) has definable Skolem functions, there is an \( \mathcal{M} \)-definable function \( g(x_1, \ldots, x_n) \) such that

\[
\mathcal{M} \models \forall x_1 \cdots \forall x_n \exists w \psi(w, x_1, \ldots, x_n) \rightarrow \psi(g(x_1, \ldots, x_n), x_1, \ldots, x_n).
\]

Therefore, if

\[
\{(x, y) \in M^2 : \mathcal{M} \models \phi(f_1(x, y), \ldots, f_n(x, y)) \} \in U^2,
\]

then \(\{(x, y) \in M^2 : \mathcal{M} \models \psi(g(f_1(x, y), \ldots, f_n(x, y)), f_1(x, y), \ldots, f_n(x, y)) \} \in U^2.\)

On the other hand, since \( h(x, y) = g(f_1(x, y), \ldots, f_n(x, y)) \) is \( \mathcal{M} \)-definable, we have

\[
\mathcal{M}^* \models \psi([h(0, 1)], [f_1(0, 1)], \ldots, [f_n(0, 1)]),
\]

and so \( \mathcal{M}^* \models \phi([f_1(0, 1)], \ldots, [f_n(0, 1)]) \).

The theorem now follows by induction. \( \square \)

(b). We need to show that \([0], [1] \) forms a set of order indiscernibles in \( \mathcal{M}^* \) over \( \mathcal{M} \).

Let \( \phi(x_0, \ldots, x_n) \) be an \( L(\mathcal{M}) \)-formula. Let \( m_1, \ldots, m_n \in M \). For each \( 1 \leq i \leq n, m_i \) is represented in \( \mathcal{M}^* \) by the constant function \( h_i(x, y) = m_i \). Let

\[
Z = \{w \in M : M \models \phi(w, m_1, \ldots, m_n)\}.
\]

Note that

\[
\{m \in M : (M \times Z) \models m \in U\} = \begin{cases} 
\emptyset & \text{if } Z \notin U \\
M & \text{if } Z \in U
\end{cases}
\]

and \( \{m \in M : (Z \times M) \models m \in U\} = Z \).

Therefore, \( M \times Z \in U^2 \) if and only if \( Z \times M \in U^2. \) Now, using part (a),

\[
\mathcal{M}^* \models \phi([0], [h_1(0, 1)], \ldots, [h_n(0, 1)]) \text{ if and only if } \\
\{(x, y) \in M^2 : \mathcal{M} \models \phi(x, m_1, \ldots, m_n)\} = Z \times M \in U^2 \text{ if and only if } \\
\{(x, y) \in M^2 : \mathcal{M} \models \phi(x, m_1, \ldots, m_n)\} = Z \times M \in U^2.
\]

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\[ \{ (x, y) \in M^2 : M \models \phi(y, m_1, \ldots, m_n) \} = M \times Z \in \mathcal{U}^2 \text{ if and only if} \]
\[ \mathcal{M}^* \models \phi([1], [h_1(0,1)], \ldots, [h_n(0,1)]). \]
This shows that \{[0], [1]\} forms a set of order indiscernibles in \( \mathcal{M}^* \) over \( \mathcal{M} \).

**3.9. Exercise.** Let \( \mathcal{U} \) be an \( \mathcal{M} \)-amenable ultrafilter. Let \((I, <)\) be an ordered set disjoint from \( M \). Let \( \mathcal{M}^* = \text{Ult}(\mathcal{M}, \mathcal{U}, I) \) (defined on p.16-17). We need to define the interpretations of the relation symbols in \( \mathcal{L}(\mathcal{M}) \) in \( \mathcal{M}^* \). Let \( R(x_1, \ldots, x_n) \) be an \( n \)-ary relation symbol in \( \mathcal{L}(\mathcal{M}) \). For all \([f_1(I_1)], \ldots, [f_n(I_n)] \in M^* \), define
\[ R^{\mathcal{M}^*}([f_1(I_1)], \ldots, [f_n(I_n)]) \text{ if and only if} \]
\[ \{ u \in M^I : M \models R(f_1([u]), \ldots, f_n([u])) \} \in \mathcal{U}^I \text{ where } I_0 = \bigcup_{1 \leq j \leq n} I_j. \]

Let \( \varepsilon : M \rightarrow M^* \) be the map defined by \( m \mapsto [f_m(I_0)] \) where \( f_m \) is the constant map with value \( m \) and \( I_0 \subseteq I \) is finite. We need to verify that \( \varepsilon \) is an isomorphism between \( \mathcal{M} \) and \( \varepsilon(\mathcal{M}) \). Let \( m_1, m_2 \in M \). We have
\[ \varepsilon(m_1) = \varepsilon(m_2) \text{ if and only if} \]
\[ \{ u \in M^I : M \models f_{m_1}(I_0) = f_{m_2}(I_1) \} \in \mathcal{U}^I \text{ where } I_0, I_1 \subseteq I \text{ are finite and } I_2 = I_0 \cup I_1 \]
\[ \text{if and only if } m_1 = m_2. \]
This shows that \( \varepsilon \) is a well-defined injection. Let \( R(x_1, \ldots, x_n) \) be an \( n \)-ary relation symbol in \( \mathcal{L}(\mathcal{M}) \). Let \( m_1, \ldots, m_n \in M \). We have
\[ \mathcal{M}^* \models R(\varepsilon(m_1), \ldots, \varepsilon(m_n)) \text{ if and only if} \]
\[ \{ u \in M^I : M \models R(f_{m_1}(I_1)[u], \ldots, f_{m_n}(I_n)[u]) \} \in \mathcal{U}^I \text{ where } I_1, \ldots, I_n \subseteq I \text{ are finite and } I_0 = \bigcup_{1 \leq j \leq n} I_j \]
\[ \text{if and only if } \mathcal{M} \models R(m_1, \ldots, m_n). \]

Let \( g(x_1, \ldots, x_n) \) be an \( n \)-ary function symbol in \( \mathcal{L}(\mathcal{M}) \). Let \( m_1, \ldots, m_n \in M \). Let \( m_0 \in M \) be such that
\[ \mathcal{M} \models m_0 = g(m_1, \ldots, m_n). \]
Now,
\[ g^{\mathcal{M}^*}(\varepsilon(m_1), \ldots, \varepsilon(m_n)) = g^{\mathcal{M}^*}([f_{m_1}(I_1)], \ldots, [f_{m_n}(I_n)]) = [h(I_0)], \]
where \( I_1, \ldots, I_n \subseteq I \) are finite and \( I_0 = \bigcup_{1 \leq j \leq n} I_j \), and \( h \) is a parametrically definable function such that for all \( u \in M^I \),
\[ h(I_0)[u] = g^M(f_{m_1}(I_1), \ldots, f_{m_n}(I_n)) = g^M(m_1, \ldots, m_n) = m_0. \]

Therefore \( h(I_0) = f_{m_0}(I_0) = \varepsilon(m_0) \). This shows that \( \varepsilon \) is an isomorphism.

**3.10. Exercise.** Let \( I_0 \subseteq I \) be finite. We show that if \( I_0 = I_1 \cup I_2 \) with \( \text{max } I_1 < \text{min } I_2 \), then for all \( X \subseteq M^I \),
\[ X \in \mathcal{U}^I \text{ if and only if } \{ s \in M^I : X|s \in \mathcal{U}^I \} \in \mathcal{U}^I. \]
by induction on the size of $I_1$. When $|I_1| = 1$ the result follows immediately from the definition of $U^{I_0}$. Suppose that the result holds for all finite $J_0 = J_1 \cup J_2 \subseteq I$ with $\max J_1 < \min J_2$ and $|J_1| = n$. Let $K_0 \subseteq I$ and suppose $K_0 = K_1 \cup K_2$ with $\max K_1 < \min K_2$ and $|K_1| = n + 1$. Suppose $K_1 = \{i_0 < \cdots < i_n\}$, and let $K'_1 = \{i_1 < \cdots < i_n\}$. Let $X \subseteq M^{K_0}$. From the definition of $X|s$, we get

$$\{s \in M^{K_1} : X|s \in U^{K_2}\} \in U^{K_1}$$

if and only if

$$\{q \in M^{K_1} : q = s \cup t \text{ where } s \in M^{(i_0)} \text{ and } t \in M^{K'_1}, (X|s)t \in U^{K_2}\} \in U^{K_1}.$$

Therefore the definition of $U^{K_1}$ yields

$$\{s \in M^{K_1} : X|s \in U^{K_2}\} \in U^{K_1}$$

if and only if

$$\{s \in M^{(i_0)} : t \in M^{K'_1} : (X|s)t \in U^{K_2}\} \in U^{(i_0)}.$$

Which, by the induction hypothesis, gives

$$\{s \in M^{K_1} : X|s \in U^{K_2}\} \in U^{K_1}$$

if and only if

$$\{s \in M^{(i_0)} : X|s \in U^{K'_1 \cup K_2}\} \in U^{(i_0)}.$$

And finally, the definition of $U^{K_0}$ gives

$$\{s \in M^{K_1} : X|s \in U^{K_2}\} \in U^{K_1}$$

if and only if $X \in U^{K_0}$.

This completes the proof.

3.12. Exercise. Let $I_2 \subseteq I$ be finite with $I_2 = I_0 \cup I_1$. Let $I \subseteq I$ be finite with $I_2 \subseteq J$. Let $f(I_0)$ and $g(I_1)$ be generalized terms. Let

$$Z_1 = \{u \in M^{I_2} : f(I_0)[u] = g(I_1)[u]\} \text{ and } Z_2 = \{u \in M^I : f(I_0)[u] = g(I_1)[u]\}.$$

We need to show that $Z_1 \in U^{I_2}$ if and only if $Z_2 \in U^I$. But this follows immediately from Lemma 3.11, since

$$Z_2 = \{s \cup t \in M^I : s \in Z_1 \land t \in M^{I \backslash I_2}\}.$$


Theorem 0.2 (Loś Theorem) Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}(M)$-formula. For all $[f_1(I_1)], \ldots, [f_n(I_n)] \in M^*$,

$$\mathcal{M}^* \models \phi([f_1(I_1)], \ldots, [f_n(I_n)])$$

if and only if

$$\{u \in M^{I_0} : \mathcal{M} \models \phi(f_1(I_1)[u], \ldots, f_n(I_n)[u])\} \in U^{I_0} \text{ where } I_0 = \bigcup_{1 \leq j \leq n} I_j.$$

Proof We prove this theorem by structural induction on $\phi$. Without loss of generality we may assume that $\phi$ only contains the logical connectives $\neg$ and $\land$, and the quantifier $\exists$. It follows from the definition of $\mathcal{M}^*$ in Subsection ?? and Exercise 3.9 that the theorem holds for all atomic formulae. As was the case with the Loś Theorem proved in Exercise 3.8, the inductive steps where $\phi = \neg \psi$ and $\phi = \psi \land \theta$ follow from the basic properties of the ultrafilter $U^{I_0}$. Again, the nontrivial inductive step involves
dealing with the quantifier $\exists$. Suppose that the theorem holds for $\psi(w, x_1, \ldots, x_n)$, and $\phi(x_1, \ldots, x_n) = \exists w \psi(w, x_1, \ldots, x_n)$. Let $[f_1(I_1), \ldots, [f_n(I_n)] \in M^*$. If
\[
\mathcal{M}^* \models \phi([f_1(I_1)], \ldots, [f_n(I_n)]),
\]
then there exists $[g(J)] \in M^*$ such that
\[
\mathcal{M}^* \models \psi([g(J)], [f_1(I_1)], \ldots, [f_n(I_n)]).
\]
Therefore, by the induction hypothesis,
\[
Z_1 = \{ u \in M^{J'} : \mathcal{M} \models \psi(g(J)[u], f_1(I_1)[u], \ldots, f_n(I_n)[u]) \in \mathcal{U}^{J'} \}
\]
where $J' = J \cup \bigcup_{1 \leq j \leq n} I_j$.

Let $I_0 = \bigcup_{1 \leq j \leq n} I_j$. Let
\[
Z_2 = \{ u \in M^{I_0} : \mathcal{M} \models \exists w \psi(w, f_1(I_1)[u], \ldots, f_n(I_n)[u]) \}
\]
and
\[
Z_3 = \{ u \in M^{J'} : \mathcal{M} \models \exists w \psi(w, f_1(I_1)[u], \ldots, f_n(I_n)[u]) \}.
\]
Now, $Z_1 \subseteq Z_3$, and so $Z_3 \in \mathcal{U}^{J'}$. And
\[
Z_3 = \{ s \cup t \in M^{J'} : s \in Z_2 \land t \in M^{J' \setminus I_0} \},
\]
so, by Lemma 3.11, $Z_2 \in \mathcal{U}^{I_0}$.

We are left to show the converse. Since $\mathcal{M}$ has definable Skolem functions, there exists a function $g(x_1, \ldots, x_n)$ such that
\[
\mathcal{M} \models \forall x_1 \cdots \forall x_n (\exists w \psi(w, x_1, \ldots, x_n) \rightarrow \psi(g(x_1, \ldots, x_n), x_1, \ldots, x_n))
\]
Let $I_0 = \bigcup_{1 \leq j \leq n} I_j$. Now, if
\[
\{ u \in M^{I_0} : \mathcal{M} \models \phi(f_1(I_1)[u], \ldots, f_n(I_n)[u]) \} \in \mathcal{U}^{I_0},
\]
then $\{ u \in M^{I_0} : \mathcal{M} \models \psi(g(f_1(I_1)[u], \ldots, f_n(I_n)[u]), f_1(I_1)[u], \ldots, f_n(I_n)[u]) \} \in \mathcal{U}^{I_0}$.

Let $h(I_0) = g(f_1(I_1), \ldots, f_n(I_n))$. Therefore
\[
\{ u \in M^{I_0} : \mathcal{M} \models \psi(h(I_0)[u], f_1(I_1)[u], \ldots, f_n(I_n)[u]) \} \in \mathcal{U}^{I_0}.
\]
And so, by the induction hypothesis,
\[
\mathcal{M}^* \models \psi([h(I_0)], [f_1(I_1)], \ldots, [f_n(I_n)]),
\]
which means $\mathcal{M}^* \models \phi([f_1(I_1)], \ldots, [f_n(I_n)])$.

The theorem now follows by induction. \qed
We now turn to showing that the embedding \( \varepsilon : \mathcal{M} \rightarrow \mathcal{M}^* \) defined in Subsection ?? is elementary. As we did in the solution to Exercise 3.9, if \( I_0 \subseteq I \) is finite and \( m \in M \), then we write \( f_m(I_0) \) for the constant function with value \( m \). Let \( \phi(x_1, \ldots, x_n) \) be an \( \mathcal{L}(\mathcal{M}) \)-formula. Let \( I_0 \subseteq I \) be finite. Note that, for all \( m_1, \ldots, m_n \in M \),

\[
\{ u \in M^{I_0} : M \models \phi(f_{m_1}(I_0)[u], \ldots, f_{m_n}(I_0)[u]) \} = \begin{cases} M^{I_0} & \text{if } M \models \phi(m_1, \ldots, m_n) \\ \emptyset & \text{otherwise} \end{cases}
\]

Therefore the Loś Theorem proved above yields:

\[
\mathcal{M}^* \models \phi(\varepsilon(m_1), \ldots, \varepsilon(m_n)) \text{ if and only if } \{ u \in M^{I_0} : M \models \phi(f_{m_1}(I_0)[u], \ldots, f_{m_n}(I_0)[u]) \} \in \mathcal{U}^{I_0} \text{ if and only if } \mathcal{M} \models \phi(m_1, \ldots, m_n).
\]

3.14. Exercise. We need to show that \((I, <)\) is a set of order indiscernibles over \( \mathcal{M} \). Let \( \phi(y_1, \ldots, y_n, x_0, \ldots, x_{k-1}) \) be an \( \mathcal{L}(\mathcal{M}) \)-formula. Let \( m_1, \ldots, m_n \in M \), and let \( i_0 < \cdots < i_{k-1} \) and \( j_0 < \cdots < j_{k-1} \) be in \( I \). Let \( I_0 = \{i_0 < \cdots < i_{k-1}\} \) and let \( J_0 = \{j_0 < \cdots < j_{k-1}\} \). Now,

\[
\mathcal{M}^* \models \phi(\varepsilon(m_1), \ldots, \varepsilon(m_n), [i_0], \ldots, [i_{k-1}]) \text{ if and only if } \{ u \in M^{I_0} : M \models \phi(m_1, \ldots, m_n, i_0[u], \ldots, i_{k-1}[u]) \} \in \mathcal{U}^{I_0} \text{ if and only if } \{ (x_0, \ldots, x_{k-1}) \in M^k : M \models \phi(m_1, \ldots, m_n, x_0, \ldots, x_{k-1}) \} \in \mathcal{U}^k \text{ if and only if } \{ u \in M^{I_0} : M \models \phi(m_1, \ldots, m_n, j_0[u], \ldots, j_{k-1}[u]) \} \in \mathcal{U}^{I_0} \text{ if and only if } \mathcal{M}^* \models \phi(\varepsilon(m_1), \ldots, \varepsilon(m_n), [j_0], \ldots, [j_{k-1}]),
\]

where the first and last equivalences follow from the Loś Theorem, and the middle two equivalences follow from the definition of the dimensional ultrapower.