A Little Blocked Literal Goes a Long Way

Benjamin Kiesl Marijn J.H. Heule Martina Seidl

TU Wien UT Austin JKU Linz

Supported by the Austrian Science Fund (FWF) under project W1255-N23.

Overview

- Topic: Proofs for quantified Boolean formulas (QBFs).

Overview

- Topic: Proofs for quantified Boolean formulas (QBFs).
- Brief overview of QBF and corresponding proof systems.

Overview

- Topic: Proofs for quantified Boolean formulas (QBFs).
- Brief overview of QBF and corresponding proof systems.
- Main result: QRAT simulates long-distance resolution.

Overview

- Topic: Proofs for quantified Boolean formulas (QBFs).
- Brief overview of QBF and corresponding proof systems.
- Main result: QRAT simulates long-distance resolution.
  - QRAT is the QBF generalization of DRAT.
  - Simulation is polynomial.

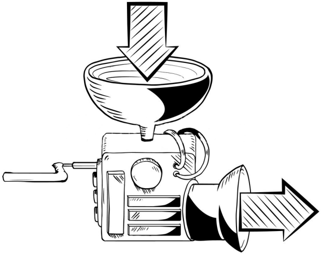
Overview

- Topic: Proofs for quantified Boolean formulas (QBFs).
- Brief overview of QBF and corresponding proof systems.
- Main result: QRAT simulates long-distance resolution.
  - QRAT is the QBF generalization of DRAT.
  - Simulation is polynomial.
- We have an implementation and evaluation of the simulation.

Satisfiability of Quantified Boolean Formulas (QSAT)

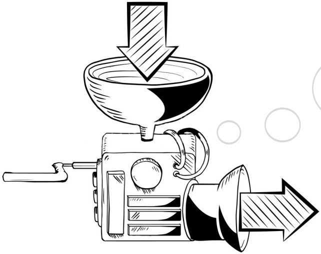
"For every truth value of x,
does there exist a truth value of y,
such that . . . "

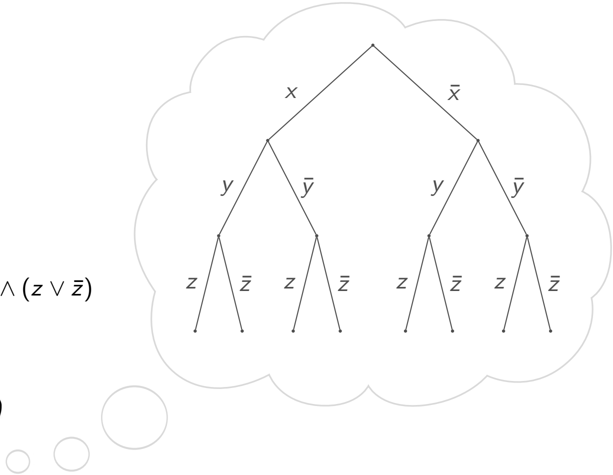
$\forall x \exists y \forall z \; (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



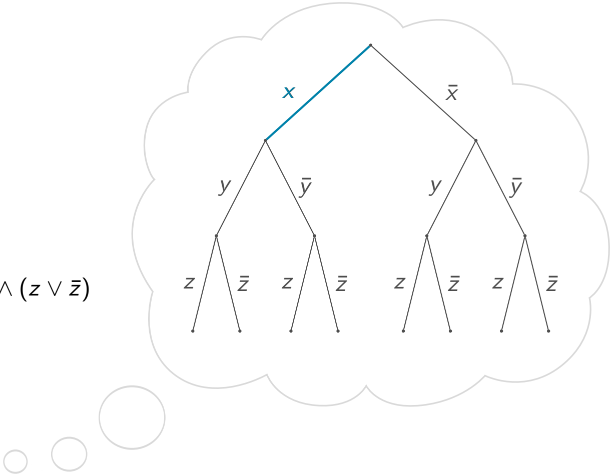
Satisfiability of Quantified Boolean Formulas (QSAT)

$\forall x \exists y \forall z \ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$

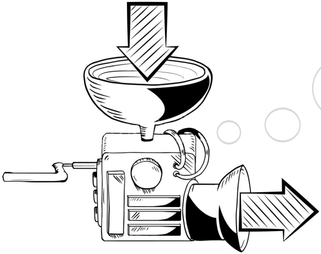




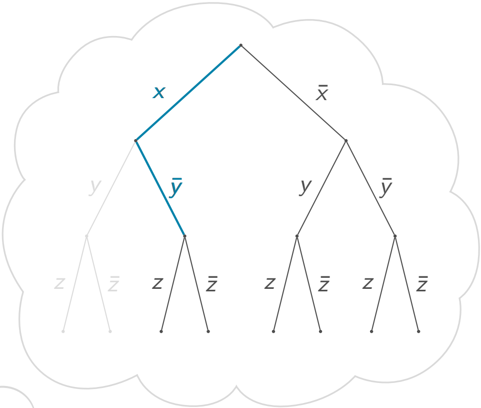
Satisfiability of Quantified Boolean Formulas (QSAT)



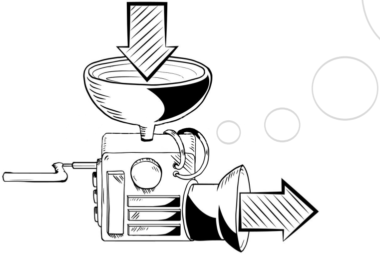
$\forall x \exists y \forall z\ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



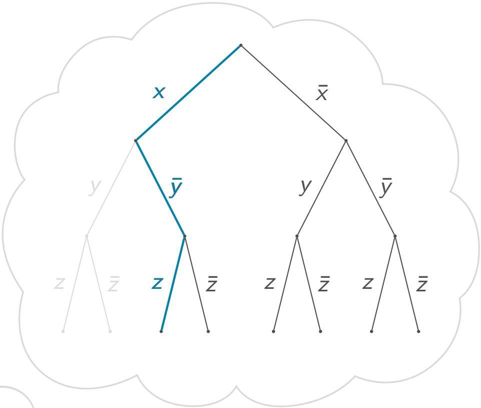
Satisfiability of Quantified Boolean Formulas (QSAT)



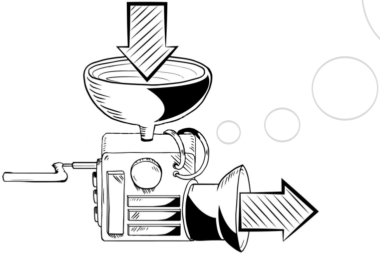
$\forall x \exists y \forall z \; (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



Satisfiability of Quantified Boolean Formulas (QSAT)

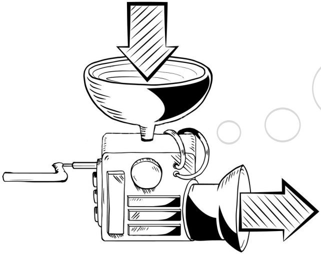


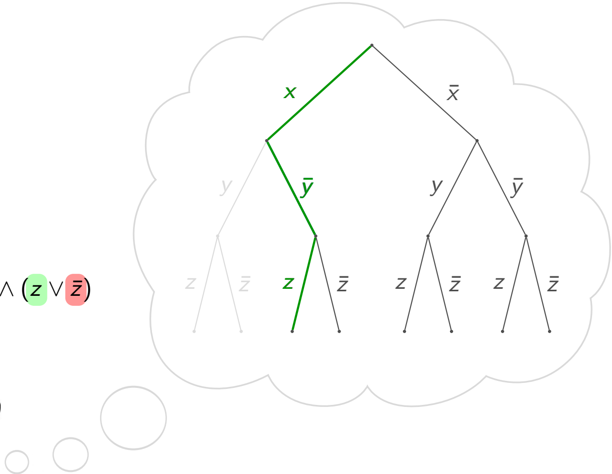
$\forall x \exists y \forall z\ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



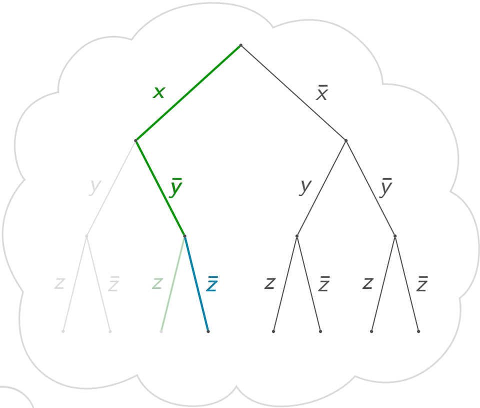
Satisfiability of Quantified Boolean Formulas (QSAT)

$\forall x \exists y \forall z\ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$

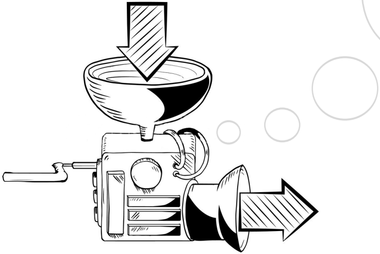




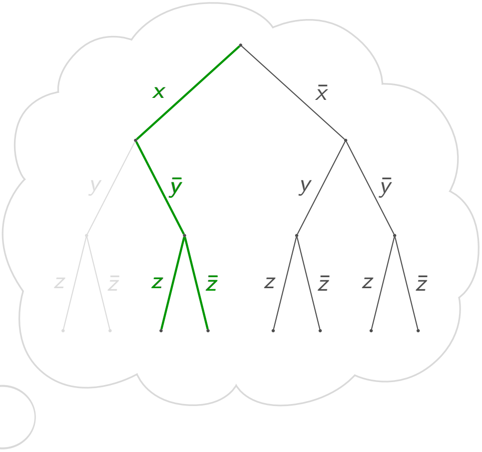
Satisfiability of Quantified Boolean Formulas (QSAT)



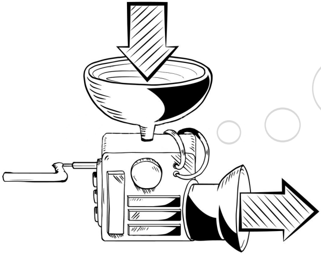
$\forall x \exists y \forall z \ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



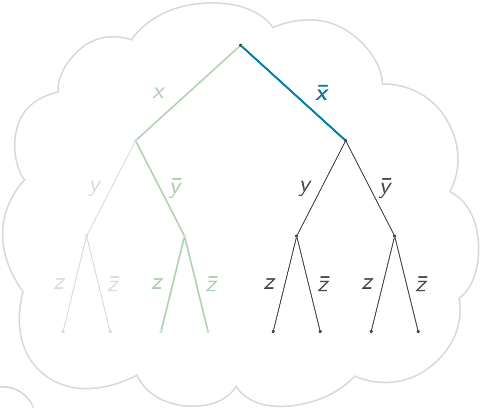
Satisfiability of Quantified Boolean Formulas (QSAT)



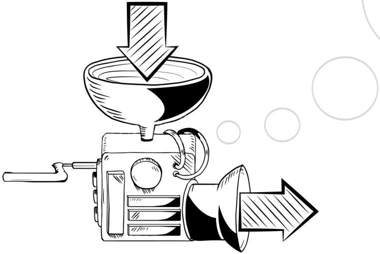
$\forall x \exists y \forall z \ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



Satisfiability of Quantified Boolean Formulas (QSAT)

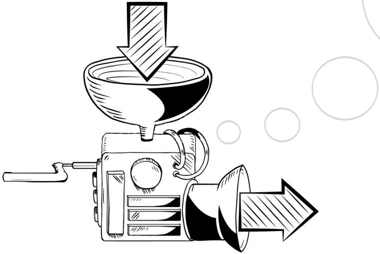


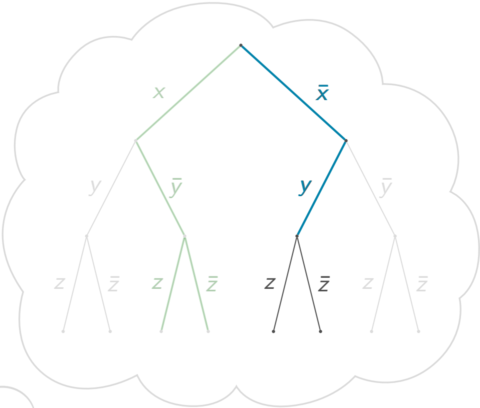
$\forall x \exists y \forall z\ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



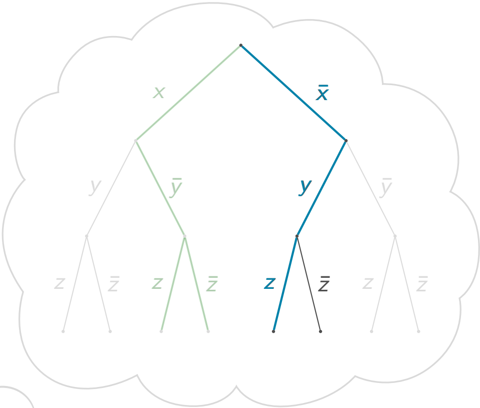
Satisfiability of Quantified Boolean Formulas (QSAT)

$\forall x \exists y \forall z \; (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$

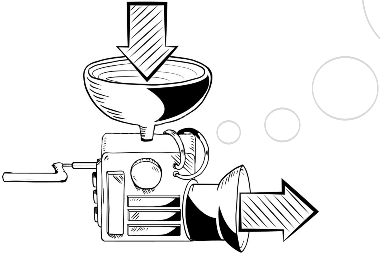




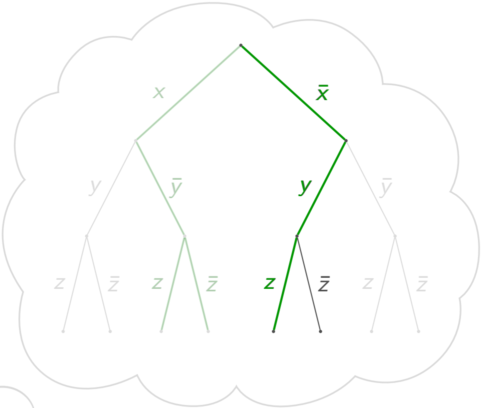
Satisfiability of Quantified Boolean Formulas (QSAT)



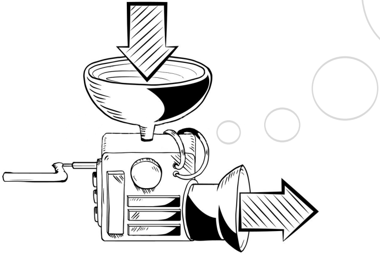
$\forall x \exists y \forall z \ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



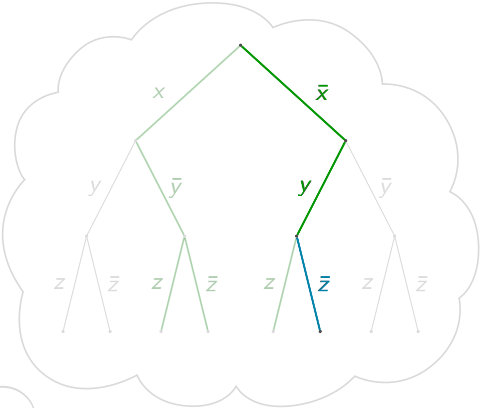
Satisfiability of Quantified Boolean Formulas (QSAT)



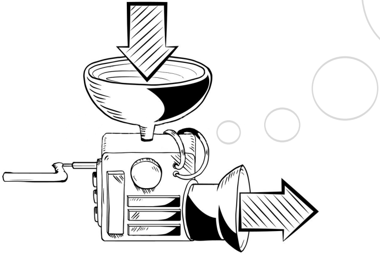
$\forall x \exists y \forall z \ (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



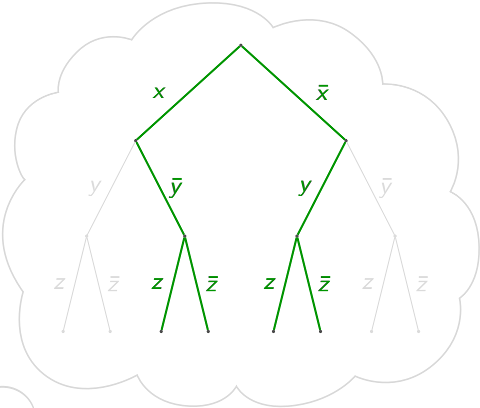
Satisfiability of Quantified Boolean Formulas (QSAT)



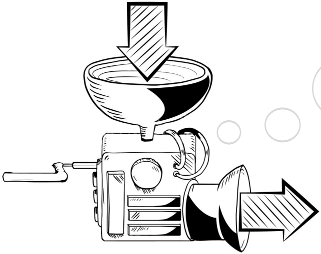
$\forall x \exists y \forall z \; (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



Satisfiability of Quantified Boolean Formulas (QSAT)



$\forall x \exists y \forall z \; (x \vee y) \wedge (\bar{x} \vee \bar{y}) \wedge (z \vee \bar{z})$



Satisfiable

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.
- Popular system: long-distance resolution (LQ-Res)
  - Perfect for search-based solving.

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.
- Popular system: long-distance resolution (LQ-Res)
  - Perfect for search-based solving.
- Alternative approach: QRAT (QBF variant of DRAT)
  - Perfect for certifying correctness of preprocessing.

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.
- Popular system: long-distance resolution (LQ-Res)
  - Perfect for search-based solving.
- Alternative approach: QRAT (QBF variant of DRAT)
  - Perfect for certifying correctness of preprocessing.
- Open question: If there is a short LQ-Res proof of a QBF, is there also a short QRAT proof?

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.
- Popular system: long-distance resolution (LQ-Res)
  - Perfect for search-based solving.
- Alternative approach: QRAT (QBF variant of DRAT)
  - Perfect for certifying correctness of preprocessing.
- Open question: If there is a short LQ-Res proof of a QBF, is there also a short QRAT proof?
  - Short = polynomial with respect to the size of the formula.

Proof Systems for QBF: LQ-Res vs. QRAT

- There exist various proof systems for QBF.
- Popular system: long-distance resolution (LQ-Res)
  - Perfect for search-based solving.
- Alternative approach: QRAT (QBF variant of DRAT)
  - Perfect for certifying correctness of preprocessing.
- Open question: If there is a short LQ-Res proof of a QBF, is there also a short QRAT proof?
  - Short = polynomial with respect to the size of the formula.
  - Our answer: Yes!

Simulating LQ-Res With QRAT

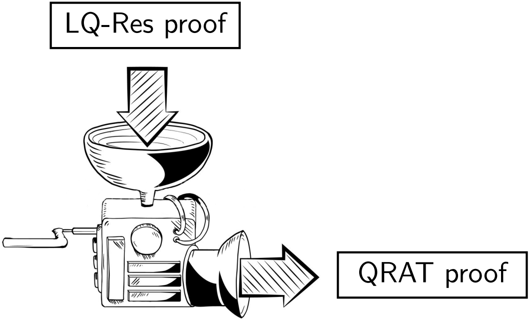
- How to show that there is a short QRAT proof for every short LQ-Res proof?

Simulating LQ-Res With QRAT

- How to show that there is a short QRAT proof for every short LQ-Res proof?

➥ Answer: With a simulation procedure.

  - Takes as input an LQ-Res proof and transforms it into a short QRAT proof.



Proving Unsatisfiability of QBFs: Long-Distance Resolution

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C} \; (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D} \; (\text{LQ-Res})$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for ∀-red: u must be universal and right of every existential literal.

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for ∀-red: u must be universal and right of every existential literal.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee e_2 \vee u_2}{e_1 \vee e_2}\ (\forall\text{-red}) \qquad ✓$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for ∀-red: u must be universal and right of every existential literal.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee e_2 \vee u_2}{e_1 \vee e_2}\ (\forall\text{-red}) \quad ✓ \qquad \frac{e_1 \vee u_1}{e_1}\ (\forall\text{-red}) \quad ✓$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for ∀-red: u must be universal and right of every existential literal.

- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee e_2 \vee u_2}{e_1 \vee e_2}\ (\forall\text{-red}) \quad ✓ \qquad \frac{e_1 \vee u_1}{e_1}\ (\forall\text{-red}) \quad ✓$$

$$\frac{e_1 \vee u_1 \vee e_2}{e_1 \vee e_2}\ (\forall\text{-red}) \quad ✗$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Each clause in an LQ-Res proof is either contained in the formula or derived via one of the following two rules:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for ∀-red: u must be universal and right of every existential literal.

- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee e_2 \vee u_2}{e_1 \vee e_2}\ (\forall\text{-red}) \quad ✓ \qquad\qquad \frac{e_1 \vee u_1}{e_1}\ (\forall\text{-red}) \quad ✓$$

$$\frac{e_1 \vee u_1 \vee e_2}{e_1 \vee e_2}\ (\forall\text{-red}) \quad ✗ \qquad\qquad \frac{e_1 \vee u_1 \vee e_2}{e_1 \vee u_1}\ (\forall\text{-red}) \quad ✗$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C} \ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D} \ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C} \ (\forall\text{-red}) \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D} \ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_1}{u_1 \vee \bar{u}_1} \ (\text{LQ-Res}) \quad ✓$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_1}{u_1 \vee \bar{u}_1}\ (\text{LQ-Res})\ ✓ \qquad \frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_2}{u_1 \vee \bar{u}_2}\ (\text{LQ-Res})\ ✓$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad\qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_1}{u_1 \vee \bar{u}_1}\ (\text{LQ-Res}) \ ✓ \qquad \frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_2}{u_1 \vee \bar{u}_2}\ (\text{LQ-Res}) \ ✓$$

$$\frac{e_1 \vee e_2 \qquad \bar{e}_1 \vee \bar{e}_2}{e_2 \vee \bar{e}_2}\ (\text{LQ-Res}) \ ✗$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Rules of LQ-Res:

$$\frac{C \vee u}{C}\ (\forall\text{-red}) \qquad \frac{C \vee l \qquad D \vee \bar{l}}{C \vee D}\ (\text{LQ-Res})$$

- Precondition for LQ-Res: (1) l is existential and (2) if there is an $x \in C$ such that $\bar{x} \in D$, then x is universal and right of l.
- Examples ($\exists e_1 \forall u_1 \exists e_2 \forall u_2$):

$$\frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_1}{u_1 \vee \bar{u}_1}\ (\text{LQ-Res})\ ✓ \qquad \frac{e_1 \vee u_1 \qquad \bar{e}_1 \vee \bar{u}_2}{u_1 \vee \bar{u}_2}\ (\text{LQ-Res})\ ✓$$

$$\frac{e_1 \vee e_2 \qquad \bar{e}_1 \vee \bar{e}_2}{e_2 \vee \bar{e}_2}\ (\text{LQ-Res})\ ✗ \qquad \frac{u_1 \vee e_2 \qquad \bar{u}_1 \vee \bar{e}_2}{u_1 \vee \bar{u}_1}\ (\text{LQ-Res})\ ✗$$

Proving Unsatisfiability of QBFs: Long-Distance Resolution

- Example proof with long-distance resolution:

  $\phi = \exists e_1 \forall u_1 \exists e_2 \exists e_3 . (\bar{e}_1 \lor \bar{u}_1 \lor e_3) \land (\bar{u}_1 \lor e_2 \lor \bar{e}_3) \land (e_1 \lor u_1 \lor e_2) \land (\bar{e}_2)$

$$
\cfrac{\cfrac{\cfrac{\cfrac{e_1 \lor u_1 \lor e_2 \qquad \cfrac{\bar{u}_1 \lor e_2 \lor \bar{e}_3 \qquad \bar{e}_1 \lor \bar{u}_1 \lor e_3}{\bar{e}_1 \lor \bar{u}_1 \lor e_2}\ (\text{LQ-res})}{u_1 \lor \bar{u}_1 \lor e_2}\ (\text{LQ-res}) \qquad \bar{e}_2}{u_1 \lor \bar{u}_1}\ (\text{LQ-res})}{u_1}\ (\forall\text{-red})}{\bot}\ (\forall\text{-red})
$$

Proving Unsatisfiability of QBFs: QRAT

- A QRAT proof is a sequence of formula modifications:
  - Add or remove so-called QRAT clauses.
  - Add or remove so-called QRAT literals.
  - ∀-reduction of non-complementary literals.

Proving Unsatisfiability of QBFs: QRAT

- A QRAT proof is a sequence of formula modifications:
  - Add or remove so-called QRAT clauses.
  - Add or remove so-called QRAT literals.
  - $\forall$-reduction of non-complementary literals.
- A formula is unsatisfiable iff the empty clause can be obtained.

Proving Unsatisfiability of QBFs: QRAT

- A QRAT proof is a sequence of formula modifications:
  - Add or remove so-called QRAT clauses.
  - Add or remove so-called QRAT literals.
  - ∀-reduction of non-complementary literals.
- A formula is unsatisfiable iff the empty clause can be obtained.
- Our simulation does not need the full power of QRAT, only:
  - Resolution
  - ∀-reduction of non-complementary literals
  - Blocked-literal elimination
  - Blocked-literal addition

Proving Unsatisfiability of QBFs: QRAT

- A QRAT proof is a sequence of formula modifications:
  - Add or remove so-called QRAT clauses.
  - Add or remove so-called QRAT literals.
  - $\forall$-reduction of non-complementary literals.
- A formula is unsatisfiable iff the empty clause can be obtained.
- Our simulation does not need the full power of QRAT, only:
  - Resolution (QRAT-clause addition)
  - $\forall$-reduction of non-complementary literals
  - Blocked-literal elimination (QRAT-literal elimination)
  - Blocked-literal addition (QRAT-literal addition)

Example: QRAT proof

1.	$a_n \vee \bar{x}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
2.	$b_n \vee x_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
3.	$a_{n-1} \vee \bar{x}_{n-1} \vee \bar{b}_n \vee \bar{x}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
4.	$b_{n-1} \vee x_{n-1} \vee \bar{a}_n \vee x_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
5.	$a_{n-1} \vee \bar{x}_{n-1} \vee \bar{b}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(BLE of $\bar{x}_n$ from 3)
6.	$b_{n-1} \vee x_{n-1} \vee \bar{a}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(BLE of x_n from 4)
7.	$a_{n-1} \vee \bar{x}_{n-1} \vee x_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
8.	$b_{n-1} \vee x_{n-1} \vee \bar{x}_n \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(Q-res)
9.	$a_{n-1} \vee \bar{x}_{n-1} \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(BLE of x_n from 7)
10.	$b_{n-1} \vee x_{n-1} \vee \bar{c}_1 \vee \cdots \vee \bar{c}_{n-1}$	(BLE of $\bar{x}_n$ from 8)

$\vdots$

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on outer resolvents:
  - The outer resolvent of $C \lor l$ and $D \lor \bar{l}$ consists of all literals in C together with the literals of D that are left of $\bar{l}$.

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on outer resolvents:
  - The outer resolvent of $C \lor l$ and $D \lor \bar{l}$ consists of all literals in C together with the literals of D that are left of $\bar{l}$.
  - Example ($\exists c_1 \exists d_1 \exists l \exists c_2 \exists d_2$): $\dfrac{c_1 \lor l \lor c_2 \qquad d_1 \lor \bar{l} \lor d_2}{c_1 \lor c_2 \lor d_1}$

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on outer resolvents:
  - The outer resolvent of $C \vee l$ and $D \vee \bar{l}$ consists of all literals in C together with the literals of D that are left of $\bar{l}$.
  - Example ($\exists c_1 \exists d_1 \exists l \exists c_2 \exists d_2$): $$\frac{c_1 \vee l \vee c_2 \qquad d_1 \vee \bar{l} \vee d_2}{c_1 \vee c_2 \vee d_1}$$
- A universal literal is blocked in a clause if all outer resolvents of the clause upon this literal are tautologies:

$$\bar{e}_2 \vee \bar{l} \vee u_2$$

$$e_1 \vee e_2 \vee l \qquad \bar{e}_1 \vee \bar{l} \vee \bar{u}_1$$

$$e_1 \vee \bar{u}_1 \vee e_3$$

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on outer resolvents:
  - The outer resolvent of $C \vee l$ and $D \vee \bar{l}$ consists of all literals in C together with the literals of D that are left of $\bar{l}$.
  - Example ($\exists c_1 \exists d_1 \exists l \exists c_2 \exists d_2$): $$\frac{c_1 \vee l \vee c_2 \qquad d_1 \vee \bar{l} \vee d_2}{c_1 \vee c_2 \vee d_1}$$
- A universal literal is blocked in a clause if all outer resolvents of the clause upon this literal are tautologies:

$$e_1 \vee e_2 \vee l$$

$$\bar{e}_2 \vee \bar{l} \vee u_2$$

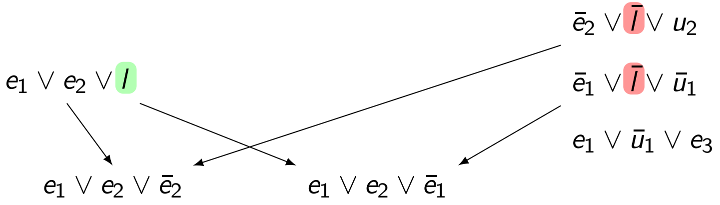
$$\bar{e}_1 \vee \bar{l} \vee \bar{u}_1$$

$$e_1 \vee \bar{u}_1 \vee e_3$$

$$e_1 \vee e_2 \vee \bar{e}_2$$

Central Concept for our Simulation: Blocked Literals

- The blocked-literal definition is based on outer resolvents:
  - The outer resolvent of $C \lor l$ and $D \lor \bar{l}$ consists of all literals in C together with the literals of D that are left of $\bar{l}$.
  - Example $(\exists c_1 \exists d_1 \exists l \exists c_2 \exists d_2)$: $$\frac{c_1 \lor l \lor c_2 \qquad d_1 \lor \bar{l} \lor d_2}{c_1 \lor c_2 \lor d_1}$$
- A universal literal is blocked in a clause if all outer resolvents of the clause upon this literal are tautologies:



Main Idea Behind the Simulation

- Problem: $\forall$-red of QRAT cannot remove complementary literals:

$$\frac{e_1 \vee u_1 \vee \bar{u}_1}{e_1 \vee u_1} \; (\forall\text{-red}) \quad \Leftarrow \text{Allowed in LQ-Res but not in QRAT}$$

Main Idea Behind the Simulation

- Problem: $\forall$-red of QRAT cannot remove complementary literals:

$$\frac{e_1 \lor u_1 \lor \bar{u}_1}{e_1 \lor u_1}\ (\forall\text{-red})$$

$\Leftarrow$ Allowed in LQ-Res but not in QRAT

- Solution: Get rid of resolvents with complementary literals.

$$\cfrac{\cfrac{e_1 \lor u_1 \lor e_2 \qquad \bar{e}_1 \lor \bar{u}_1 \lor e_2}{u_1 \lor \bar{u}_1 \lor e_2}\ (\text{LQ-res}) \qquad \bar{e}_2}{\cfrac{\cfrac{u_1 \lor \bar{u}_1}{u_1}\ (\forall\text{-red})}{\bot}\ (\forall\text{-red})}\ (\text{LQ-res})$$

Main Idea Behind the Simulation

- Problem: ∀-red of QRAT cannot remove complementary literals:

$$\frac{e_1 \lor u_1 \lor \bar{u}_1}{e_1 \lor u_1}\,(\forall\text{-red})$$ ⇐ Allowed in LQ-Res but not in QRAT

- Solution: Get rid of resolvents with complementary literals.

$$\frac{e_1 \lor u_1 \lor e_2 \qquad \bar{e}_1 \lor \bar{u}_1 \lor e_2}{u_1 \lor \bar{u}_1 \lor e_2}\,(\text{LQ-res})$$

Main Idea Behind the Simulation

- Problem: ∀-red of QRAT cannot remove complementary literals:

$$\frac{e_1 \lor u_1 \lor \bar{u}_1}{e_1 \lor u_1}\ (\forall\text{-red}) \quad \Leftarrow \text{Allowed in LQ-Res but not in QRAT}$$

- Solution: Get rid of resolvents with complementary literals.

$$\frac{e_1 \lor u_1 \lor e_2 \qquad \bar{e}_1 \lor \bar{u}_1 \lor e_2}{u_1 \lor \bar{u}_1 \lor e_2}\ (\text{LQ-res})$$

- The literal u_1 is a blocked literal and can be removed:
  - The outer resolvent $e_1 \lor \bar{e}_1$ of the two clauses is a tautology.

Main Idea Behind the Simulation

- Problem: $\forall$-red of QRAT cannot remove complementary literals:

$$\frac{e_1 \vee u_1 \vee \bar{u}_1}{e_1 \vee u_1}\ (\forall\text{-red}) \quad \Leftarrow \text{Allowed in LQ-Res but not in QRAT}$$

- Solution: Get rid of resolvents with complementary literals.

$$\frac{e_1 \vee u_1 \vee e_2 \qquad \bar{e}_1 \vee \bar{u}_1 \vee e_2}{u_1 \vee \bar{u}_1 \vee e_2}\ (\text{LQ-res})$$

- The literal u_1 is a blocked literal and can be removed:
  - The outer resolvent $e_1 \vee \bar{e}_1$ of the two clauses is a tautology.

$$\frac{\dfrac{e_1 \vee u_1 \vee e_2}{e_1 \vee e_2}\ (\text{BLE}) \qquad \bar{e}_1 \vee \bar{u}_1 \vee e_2}{\bar{u}_1 \vee e_2}\ (\text{LQ-res})$$

Main Idea Behind the Simulation

- In the preceding example, u_1 was a blocked literal.
- This is not always the case.

Main Idea Behind the Simulation

- In the preceding example, u_1 was a blocked literal.
- This is not always the case.
  - But, using blocked-literal addition, we can always remove complementary literals.
  - For details, see our paper.

Main Idea Behind the Simulation

- In the preceding example, u_1 was a blocked literal.
- This is not always the case.
  - But, using blocked-literal addition, we can always remove complementary literals.
  - For details, see our paper.
- By successively removing complementary literals from resolution steps, we obtain a valid QRAT proof.

Simulation Procedure: Results

- Our simulation procedure produces a QRAT proof with at most a quadratic blow-up in size.

Simulation Procedure: Results

- Our simulation procedure produces a QRAT proof with at most a quadratic blow-up in size.
- We implemented the procedure, the tool is called `ld2qrat`.

Simulation Procedure: Results

- Our simulation procedure produces a QRAT proof with at most a quadratic blow-up in size.
- We implemented the procedure, the tool is called `ld2qrat`.
  - Input: Long-distance-resolution proof in the QPR format.
  - Output: QRAT proof.

Simulation Procedure: Results

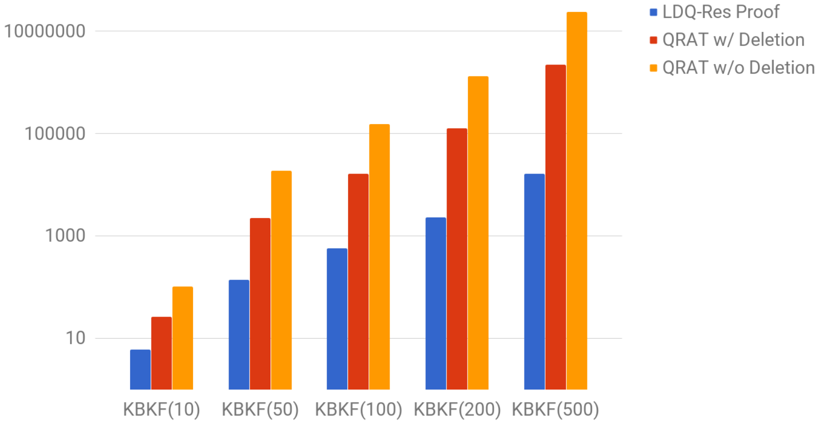
- Our simulation procedure produces a QRAT proof with at most a quadratic blow-up in size.
- We implemented the procedure, the tool is called `ld2qrat`.
  - Input: Long-distance-resolution proof in the QPR format.
  - Output: QRAT proof.
  - Several optimizations to reduce proof size (clause deletion!).

Simulation Procedure: Results

- Our simulation procedure produces a QRAT proof with at most a quadratic blow-up in size.
- We implemented the procedure, the tool is called `ld2qrat`.
  - Input: Long-distance-resolution proof in the QPR format.
  - Output: QRAT proof.
  - Several optimizations to reduce proof size (clause deletion!).
- The tool allows to merge a QRAT proof of a preprocessor with a long-distance-resolution proof of a search-based solver.

Kleine Büning Formulas (KBKF): LDQ-Res to QRAT

File size of generated proofs: LDQ-Res (Egly et al. 2013) to QRAT with and without deletion.



Simulation Procedure: Results

- Our simulation also gave insight for constructing short QRAT proofs by hand.

Simulation Procedure: Results

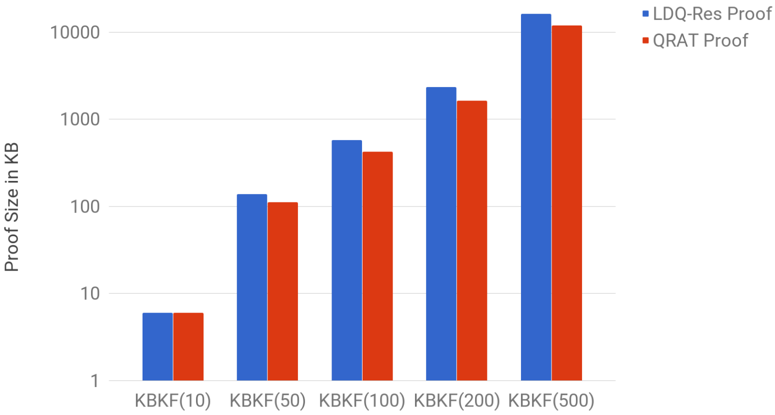
- Our simulation also gave insight for constructing short QRAT proofs by hand.
  - Formulas well-known for having short LQ-Res proofs but being hard for other proof systems: Kleine Büning formulas

Simulation Procedure: Results

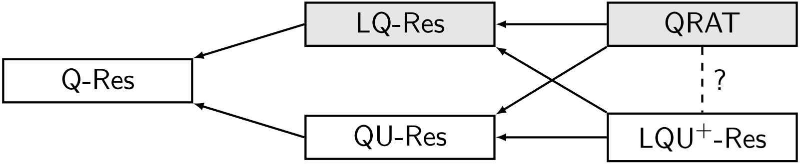
- Our simulation also gave insight for constructing short QRAT proofs by hand.
  - Formulas well-known for having short LQ-Res proofs but being hard for other proof systems: Kleine Büning formulas
  - We have hand-crafted QRAT proofs of these formulas that are shorter than the LQ-Res proofs.

Kleine Büning Formulas (KBKF): QRAT vs. LDQ-Res

File size of hand-crafted proofs: LDQ-Res (Egly et al. 2013) vs. QRAT.

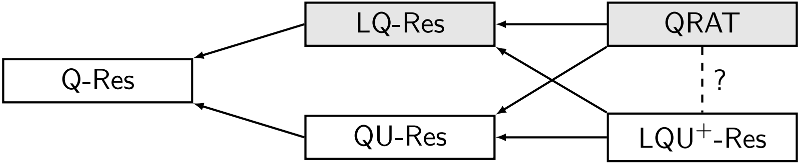


Complexity Landscape: QRAT and Resolution Systems



- Open question: Can QRAT simulate LQU$^+$-Res?
  - LQU$^+$-Res allows long-distance resolution upon universal literals.

Complexity Landscape: QRAT and Resolution Systems



- Open question: Can QRAT simulate LQU$^+$-Res?
  - LQU$^+$-Res allows long-distance resolution upon universal literals.
- Relationship between QRAT and expansion-based systems?

Conclusion

- We shed light on the relationship between LQ-Res and QRAT

Conclusion

- We shed light on the relationship between LQ-Res and QRAT
  - LQ-Res is a popular system for QBF solving.

Conclusion

- We shed light on the relationship between LQ-Res and QRAT
  - LQ-Res is a popular system for QBF solving.
  - QRAT is the best system for QBF preprocessing.

Conclusion

- We shed light on the relationship between LQ-Res and QRAT
  - LQ-Res is a popular system for QBF solving.
  - QRAT is the best system for QBF preprocessing.
- QRAT turns out to be stronger than LQ-Res.

Conclusion

- We shed light on the relationship between LQ-Res and QRAT
  - LQ-Res is a popular system for QBF solving.
  - QRAT is the best system for QBF preprocessing.
- QRAT turns out to be stronger than LQ-Res.
- Our tool allows to transform LQ-Res proofs into QRAT proofs.