Adding a Total Order to ACL2

Panagiotis Manolios[1] and Matt Kaufmann[2]

[1] College of Computing, CERCS Lab
Georgia Institute of Technology
`manolios@cc.gatech.edu`
`http://www.cc.gatech.edu/~manolios`,
[2] Advanced Micro Devices, Inc.
`matt.kaufmann@amd.com`

Abstract. We show that adding a total order to ACL2, via new axioms, allows for simpler and more elegant definitions of functions and libraries of theorems. We motivate the need for a total order with a simple example and explain how a total order can be used to simplify existing libraries of theorems (*i.e.*, ACL2 books) on finite set theory and records. These ideas have been incorporated into ACL2 Version 2.6, which includes axioms positing a total order on the ACL2 universe.

1 Introduction

ACL2 [7, 6, 8] is a logic of total functions. One particularly pleasant consequence is that many properties of functions can be stated as unconditional rewrite rules. For example, we can prove `(equal (* y (* x z)) (* x (* y z)))` without having to establish that `x`, `y`, and `z` are numbers. Such unconditional rewrite rules lead to simpler libraries of theorems, which in turn improve the ability of ACL2 to reduce large terms automatically and efficiently.

Unfortunately, it is problematic to exploit fully the totality of functions in ACL2 Version 2.5. One is often forced to use rewrite rules with hypotheses because of the lack of a definable total order on the ACL2 universe.

To examine the issue, consider the problem of sorting, *i.e.*, the problem of defining a function that returns an ordered permutation of its input. Since ACL2 is untyped, one might expect that it is possible to define a function `isort` that satisfies the following two properties (where `(perm x y)` holds iff `x` is a permutation of `y`).

```
(perm x (isort x))
(equal (perm x y)
       (equal (isort x)
              (isort y)))
```

Unfortunately, this does not seem to be the case, even if we restrict `x` and `y` to lists of length two. One is forced to write sorting functions for different types, the very thing we are trying to avoid.

The reason why a sorting function with the above properties cannot be defined is that the ACL2 universe is not closed. That is, the ACL2 axioms do not rule out the existence of *bad atoms*, atoms that do not satisfy any of `acl2-numberp`, `symbolp`, `characterp`, or `stringp`. To define a sorting function on the whole of the ACL2 universe, we need a way to order bad atoms. This is not possible since in general the set of bad atoms cannot be linearly ordered; see Section 2.3.

In the next section, we show that extending ACL2 Version 2.5 with a total order on the bad atoms is equivalent to extending it with a total order on the ACL2 universe. In the same section, we also discuss total orders from a set-theoretic point of view. We then consider three example applications where the existence of a total order is used to simplify matters. In Section 3, we define sorting functions that satisfy the above properties. In Section 4 we show how to simplify J Moore's finite set theory book [12]. In Section 5, we show how to simplify a book on records due to Kaufmann and Sumners [14, 10]. Section 6 contains the conclusions. The full proofs can be found in the supporting books, which are freely available. In this paper we often omit definitions, theorems, and parts of events (such as the `:rule-classes` portion of theorems) for presentation purposes.

2 Total Orders

We show that extending ACL2 Version 2.5 with a total order on the bad atoms is equivalent to extending ACL2 with a total order on the ACL2 universe. The full ACL2 proofs are in the book `total-order`, which is written for ACL2 Version 2.5. We also discuss total order from a set-theoretic point of view.

2.1 A Total Order for ACL2

We start by defining `bad-atom`.

```
(defun bad-atom (x)
  (not (or (consp x)
           (acl2-numberp x)
           (symbolp x)
           (characterp x)
           (stringp x))))
```

We now use `defaxiom` to introduce a total order. A relation $\preccurlyeq$ is a *total order* if it satisfies the following.

1. $x \preccurlyeq x$
2. $x \preccurlyeq y \quad \wedge \quad y \preccurlyeq x \quad \Rightarrow \quad x = y$
3. $x \preccurlyeq y \quad \wedge \quad y \preccurlyeq z \quad \Rightarrow \quad x \preccurlyeq z$
4. $x \preccurlyeq y \quad \vee \quad y \preccurlyeq x$

A *partial order* is one satisfying the first three conditions, reflexivity, antisymmetry, and transitivity, respectively. Notice that reflexivity follows from totality, the fourth condition. In the ACL2 logic, we have the following definitions.

```
(defstub bad-atom<= (* *) => *)

(defmacro boolp (x)
  `(or (equal ,x t)
       (equal ,x nil)))

(defaxiom boolp-bad-atom<=
  (boolp (bad-atom<= x y)))

(defaxiom bad-atom<=-antisymmetric
  (implies (and (bad-atom x)
                (bad-atom y)
                (bad-atom<= x y)
                (bad-atom<= y x))
           (equal x y)))

(defaxiom bad-atom<=-transitive
  (implies (and (bad-atom<= x y)
                (bad-atom<= y z)
                (bad-atom x)
                (bad-atom y)
                (bad-atom z))
           (bad-atom<= x z)))

(defaxiom bad-atom<=-total
  (implies (and (bad-atom x)
                (bad-atom y))
           (or (bad-atom<= x y)
               (bad-atom<= y x))))
```

We say that function `f` is a total order on objects recognized by `o` if `f` satisfies the properties obtained by replacing `bad-atom` with `o` and `bad-atom<=` by `f` in the four preceding properties.

We use `bad-atom<=` to define `atom-order`, a total order on atoms by using `<=` to compare numbers, `char-code` to compare characters, `string<=` to compare strings, `symbol-<` to compare symbols and `bad-atom<=` to compare everything else. Moreover, in this order numbers precede characters, which precede strings, which precede symbols, which precede bad atoms. The definition follows.

```
(defun atom-order (x y)
  (cond ((rationalp x)
         (if (rationalp y)
             (<= x y)
           t))
        ((rationalp y) nil)
        ((complex-rationalp x)
         (if (complex-rationalp y)
              (or (< (realpart x) (realpart y))
                  (and (= (realpart x) (realpart y))
                       (<= (imagpart x) (imagpart y))))
            t))
         ((complex-rationalp y)
          nil)
         ((characterp x)
          (if (characterp y)
              (<= (char-code x)
                  (char-code y))
            t))
         ((characterp y) nil)
         ((stringp x)
          (if (stringp y)
              (and (string<= x y) t)
            t))
         ((stringp y) nil)
         ((symbolp x)
          (if (symbolp y)
              (not (symbol-< y x))
            t))
         ((symbolp y) nil)
         (t (bad-atom<= x y))))
```

With the help of a few lemmas, we have that `atom-order` is a total order on atoms.

We now define a total order on the ACL2 universe as follows.

```
(defun total-order (x y)
  (cond ((atom x)
         (cond ((atom y)
                (atom-order x y))
               (t t)))
        ((atom y) nil)
        ((equal (car x) (car y))
         (total-order (cdr x) (cdr y)))
        (t (total-order (car x) (car y)))))
```

It follows directly from the definition of a total order (at the beginning of this section, and in book `total-order` in the supporting materials) that `total-order` is a total order on the ACL2 universe. We have shown that adding a total order to the bad atoms in ACL2 implies that there is a total order on the ACL2 universe. The other direction is straightforward and is given in the supporting book `total-order-easy-direction`.

2.2 Soundness

We have seen that adding axioms to ACL2 positing the existence of a total order on the whole of the ACL2 universe is equivalent to adding axioms positing the existence of a total order on the bad atoms. An important question is whether adding such axioms preserves soundness.

Soundness is preserved, as we now show. First, note that if the ACL2 axioms are sound, there are models with no bad atoms: although the axioms do not close the universe, neither do they populate it with bad atoms. Thus, if we can show that there is a total order on the ACL2 universe without bad atoms, we are done. A proof of this can be found in the supporting book `soundness`.

We define a good atom as follows.

```
(defun good-atom (x)
  (and (atom x)
       (or (acl2-numberp x)
           (symbolp x)
           (characterp x)
           (stringp x))))
```

A predicate to recognize good objects, objects in the ACL2 universe without bad atoms (*i.e.*, objects built from good-atoms), follows.

```
(defun good-object (x)
  (if (consp x)
      (and (good-object (car x))
           (good-object (cdr x)))
    (good-atom x)))
```

In the book `soundness` we exhibit a total ordering on the good objects. The definitions follow closely those in the previous section, thus we do not present them here.

Finally, we note that a somewhat stronger property holds for the axioms on total order, namely, conservativity: No additional theorems are provable from any ACL2 Version 2.5 books not involving the total order on bad atoms when the axioms on `bad-atom` are added. This is immediate from the lemma immediately preceding the theorem in Appendix B of [9], which in fact guarantees that it is conservative to add a bijection of the ACL2 universe with the natural numbers.

2.3 Total Orders From a Set-Theoretic Perspective

In this section we review some well-know facts about total orders in a purely set-theoretic setting. Good references include the books by Kunen, Devlin, and Halmos [11, 3, 4] and Part B of the Handbook of Mathematical Logic [1], especially chapter B.2 on the Axiom of Choice [5]. This section can be skipped without impacting readability of the rest of this paper.

Recall the Axiom of Choice, which (among many equivalent formulations) can be stated as follows: every set can be well-ordered. Thus, in ZFC (Zermelo-Frankel set theory with the Axiom of Choice), every set can be totally ordered.

In ZF (ZFC without the Axiom of Choice), one cannot prove that every set can be totally (linearly) ordered. One needs further axioms, but not the full power of the Axiom of Choice. For example, the Boolean Prime Ideal Theorem, a weaker statement than the Axiom of Choice, will do. The Boolean Prime Ideal Theorem (every Boolean algebra has a prime ideal) is equivalent to the Compactness Theorem of model theory: If every finite subset of S, a set of first-order sentences, has a model, then S has a model. From the compactness theorem one can prove that every set can be totally ordered.

Define C_n, for n a positive integer, to be the following statement.

> If F is a family of sets each of which has exactly n elements, then F has a choice function.

C_2 is not provable in ZF. An instructive example of this is due to Russell. Consider an infinite set of pairs of shoes. A choice function (a function that chooses one shoe from each pair) is easy to find, *e.g.*, always choosing the right shoe will do. In contrast, consider an infinite set of pairs of socks (where there is no distinction between the socks in a pair). It does not seem possible to construct a choice function, as there is no way to simultaneously choose one of the two socks for each pair. Tarski initiated the investigation of the C_n statements and showed that C_2 implies C_4. There have been various papers clarifying the relationships between combinations of C_n's, *e.g.*, it is known that C_2 implies C_n is provable iff n is 1,2, or 4.

We raised the C_n problem because there is an obvious parallel between the socks problem and the problem mentioned in the introduction, namely the problem of sorting lists of length two. In both cases, we are required to find a choice function on a collection of pairs, where there is no way to distinguish between elements in a pair.

2.4 The Total Order in ACL2 Version 2.6

We have justified adding axioms to ACL2 positing the existence of a total order. We now give an overview of the total order provided with ACL2 Version 2.6. The file `axioms.lisp` contains the definition of function `bad-atom` and axioms positing that `bad-atom<=` is a total order on the bad atoms, as above. The function `alphorder` is defined and shown to be a total order on atoms. The definition of `alphorder` is similar to our definition of `atom-order`. Finally, `lexorder`, a function corresponding to our `total-order`, is defined and shown to be a total order on the ACL2 universe. In directory `books/misc`, the ACL2 distribution for Version 2.6 contains the book `total-order`, which defines an order `<<` based on `lexorder`: `(<< x y)` if and only if `x` and `y` are not equal and `(lexorder x y)`. Having justified the addition of new axioms positing the existence of a total order, we will now use the Version 2.6 books in the sequel.

3 Sorting

We show how the new total ordering axioms can be used to sort arbitrary ACL2 objects. We then define insertion sort and quicksort and show that they are equal.

Our approach is in line with ACL2 tradition: we define total functions with an intended domain, generally the set of true (null-terminated) lists, while keeping the definitions simple by avoiding any reference to the intended domain. For the most part, we can prove theorems that do not contain assumptions about the intended domain (*i.e.*, `true-listp` hypotheses). This leads to simpler rewrite rules (see [7] for a full explanation).

3.1 Permutations and Ordered Lists

We start by defining `perm`, a function that checks if its arguments are permutations, and related theorems in the book `perm`.

```
(defun in (a X)
  (cond ((atom X) nil)
        ((equal a (car X)) t)
        (t (in a (cdr X)))))

(defun remove-el (a x)
  (cond ((atom x) nil)
        ((equal a (car x)) (cdr x))
        (t (cons (car x) (remove-el a (cdr x))))))

(defun perm (x y)
  (cond ((atom x) (atom y))
        (t (and (in (car x) y)
                (perm (cdr x) (remove-el (car x) y))))))
```

We prove that `perm` is an equivalence relation.

```
(defequiv perm)
```

This allows us to use congruence-based reasoning and we prove various congruence rules, including the following.[1]

```
(defcong perm perm (append x y) 1)
(defcong perm perm (append x y) 2)
(defcong perm perm (cons x y) 2)
(defcong perm equal (in x y) 2)
```

In the book `perm-order`, we define the function `orderedp` which checks if its argument is ordered, using the total order `lexorder`.

```
(defun <<= (x y) (lexorder x y))
```

[1] Close analogues of these results are proved with the built-in ACL2 function `mini-proveall`.

```
(defun orderedp (x)
  (cond ((atom x) (null x))
        (t (or (null (cdr x))
               (and (<<= (car x) (cadr x))
                    (orderedp (cdr x)))))))
```

3.2 Insertion Sort

In the book `insertion-sort`, we define insertion sort as follows.

```
(defun insert (a x)
  (if (consp x)
      (if (<<= a (car x))
          (cons a x)
        (cons (car x) (insert a (cdr x))))
    (list a)))

(defun isort (x)
  (if (consp x)
      (insert (car x) (isort (cdr x)))
    nil))
```

We prove the usual theorems, taking advantage of the total order to eliminate the need for hypotheses that restrict the types of the list elements.

```
(defthm ordered-sort
  (orderedp (isort x)))


(defthm perm-sort
  (perm (isort x) x))
```

We also prove the following theorem, thereby showing how to solve the problem we raised in the introduction.

```
(defthm main
  (equal (perm x y)
         (equal (isort x)
                (isort y))))
```

Two related theorems, showing that any ordered permutation of an object equals the `isort` of the object and that ordered permutations are equal follow.

```
(defthm main2
  (implies (and (orderedp x)
                (perm x y))
           (equal (isort y)
                  x)))
```

```
(defthm main3
  (implies (and (orderedp x)
                (orderedp y)
                (perm x y))
           (equal x y)))
```

3.3 Quicksort

In the book `quicksort`, we define quicksort as follows.

```
(defun less (x lst)
  (cond ((atom lst) nil)
        ((<< (car lst) x)
         (cons (car lst) (less x (cdr lst))))
        (t (less x (cdr lst)))))

(defun notless (x lst)
  (cond ((atom lst) nil)
        ((not (<< (car lst) x))
         (cons (car lst) (notless x (cdr lst))))
        (t (notless x (cdr lst)))))

(defun qsort (x)
  (cond ((atom x) nil)
        (t (append (qsort (less (car x) (cdr x)))
                   (list (car x))
                   (qsort (notless (car x) (cdr x)))))))
```

Besides proving the obvious theorems,

```
(defthm qsort-is-ordered
  (orderedp (qsort x)))
(defthm perm-qsort
  (perm (qsort x) x))
```

we also prove the following theorem, with the help of theorem `main2`, which relates quicksort and insertion sort.

```
(defthm qsort-main
  (equal (qsort x)
         (isort x))
  :hints (("goal" :use (:instance main2 (x (qsort x)) (y x)))))
```

A referee has noted that the use of a total order on the ACL2 universe can lead to code that is computationally less efficient than the use of an order on the intended domain, for example in sorting. As the referee also points out, this need not be a troublesome issue: it would be straightforward to define two sets of functions, one using the total order on the ACL2 universe and one using the domain-specific order, and then prove their correspondence on the intended domain. Then one can happily use the first set of functions for reasoning and the second set of functions for execution.

4 Set Theory

Various efforts to design ACL2 books on set theory have led researchers to define a function, call it `c`, to put sets into a canonical form. Ideally, `c` should have the property that `X` and `Y` are equal as sets iff `(equal (c X) (c Y))`. It is not possible to define such a function without restricting the set of objects that can be put into canonical form. (See the discussion of pairs of socks in Section 2.3.) In this section, we examine J Moore's book on finite set theory [12] and show how it can be simplified with the use of a total order.

J Moore's books on set theory, which can also be found in the ACL2 distribution in directory `books/finite-set-theory/`, are included in the supporting books. They are `total-ordering-original` and `set-theory-original`. The books `total-ordering` and `set-theory`, also part of the supporting books, are based on Moore's books, but use the total order. We now discuss the differences.

The original books include a definition of "ordinary" objects and "standard" atoms.

```
(defun ordinaryp (x)
  (cond ((atom x)
         (or (acl2-numberp x)
             (characterp x)
             (stringp x)
             (symbolp x)))
        (t (and (consp x)
                (ordinaryp (car x))
                (ordinaryp (cdr x))))))

(defun standard-atom (x)
  (or (acl2-numberp x)
      (characterp x)
      (stringp x)
      (symbolp x)))
```

A binary function, `<<`, is defined and shown to be a total order on ordinary objects. In addition, various functions, including `o-fix`, a function which converts objects into ordinary objects, are used to define `canonicalize`, a function that puts ordinary objects into canonical form, and `canonicalp`, a function that recognizes canonical objects.

With our books, the functions `ordinaryp`, `standard-atom`, and `o-fix` are not required and all of the theorems and definitions that depend on them can be elided or simplified. For example, the theorem

```
(defthm ordinaryp-set-insert
  (implies (and (ordinaryp e)
                (ordinaryp x))
           (ordinaryp (set-insert e x))))
```

appears in the original books but is not needed in our books. The theorem

```
(defthm <<-cons-1
  (implies (and (ordinaryp x)
                (ordinaryp y))
           (<< x (cons x y))))
```

which gives rise to a conditional rewrite rule is simplified to the following unconditional rewrite rule in our books.

```
(defthm <<-cons-1
  (<< x (cons x y)))
```

The conceptual simplifications result in better certification times. In one pair of tests the new books required 3 minutes and 4.9 seconds of user time to certify whereas the original books required 4 minutes and 21.0 seconds.

5 Records

Many different approaches to dealing with records have appeared in the work of various ACL2 users. This is not surprising, as records play an important role in programming. One approach is to use alists, where the car of an element indicates the field and the cdr indicates the value. For big projects such as the FM9801 verification effort by Sawada [13] the `structures` book by Bishop Brock has been found useful [2]. At the first ACL2 Workshop, Ken Albin gave a talk where he reviewed some of the approaches to memory modeling and requested that the community come up with a nice solution.

One problem from which all of the above approaches suffer is the complexity of the rewrite rules obtained. For example, when using alists one often uses congruence-based reasoning because there are many ways to represent the same record: permutations of the alist tend to represent the same record.

A very promising approach to dealing with records in ACL2 is due to Kaufmann and Sumners. They developed a book on records with very simple rewrite rules: specifically, there are no hypotheses that require objects to be well-formed records. Here are the main theorems proved, where `s` is a function to set a field in a record and `g` is a function to the get the value of a field in a record. The complete proof scripts are from [14] and are in the book `records-original` which is part of the supporting materials.

```
(defthm s-same-g
  (implies (force (fieldp a))
           (equal (s a (g a r) r)
                  r)))
```

```
(defthm g-same-s
  (implies (force (fieldp a))
           (equal (g a (s a v r))
                  v)))

(defthm s-same-s
  (implies (force (fieldp a))
           (equal (s a y (s a x r))
                  (s a y r))))

(defthm s-diff-s
  (implies (and (force (fieldp a))
                (force (fieldp b))
                (not (equal a b)))
           (equal (s b y (s a x r))
                  (s a x (s b y r))))
  :rule-classes ((:rewrite :loop-stopper ((b a s)))))

(defthm g-diff-s
  (implies (and (force (fieldp a))
                (force (fieldp b))
                (not (equal a b)))
           (equal (g a (s b v r))
                  (g a r))))
```

We do not give the definitions of `s` and `g` here, as they are complicated and not germane to this paper. The interesting thing about this approach is that the focus is on the algebraic properties of the functions in question. Even though the definitions of `s` and `g` are messy, it does not matter because the rewrite rules obtained are very simple and when proving theorems we only use the rewrite rules, as the definitions of `s` and `g` are disabled.

One drawback is the need for hypotheses about objects being fields. The reason for this is no surprise. We need a way to order objects and since we cannot order bad atoms, we are forced to choose a subset of the ACL2 universe and to require that objects are in this class, by adding hypotheses to our rewrite rules. When we subsequently decide to use some other type of field, we have to revise the book. For example, originally fields were symbols, then they became symbols or integers, and one can imagine many definitions of “field”. The most general hypothesis is probably that they are good objects as defined in the set theory section, above. Using the total order we can now prove the following theorems. The full proof scripts are in the supporting book `records`. Also, records books that take advantage of the total order are described in [10].

```
(defthm s-same-g
  (equal (s a (g a r) r)
         r))
```

```
(defthm g-same-s
  (equal (g a (s a v r))
         v))

(defthm s-same-s
  (equal (s a y (s a x r))
         (s a y r)))

(defthm s-diff-s
  (implies (not (equal a b))
           (equal (s b y (s a x r))
                  (s a x (s b y r))))
  :rule-classes ((:rewrite :loop-stopper ((b a s)))))

(defthm g-diff-s
  (implies (not (equal a b))
           (equal (g a (s b v r))
                  (g a r))))
```

As before, we get a non-trivial reduction in the both the size of the resulting books and the certification time. The size of the original records book is 382 lines, whereas our book is 288 lines. In addition, the certification time was reduced from 74 seconds to 33 seconds. Perhaps more important is the elimination of hypotheses, resulting in more efficient rewrite rules.

6 Conclusion

We presented new ACL2 axioms positing the existence of a total order on the ACL2 universe. These axioms have been added to ACL2 Version 2.6. We showed that a total order often simplifies ACL2 books and leads to simpler and more efficient rewrite rules. We applied the ideas to develop books on sorting and to simplify existing books on finite set theory and records.

Acknowledgements

We received many useful comments from Rob Sumners and the rest of the Austin ACL2 group. We also thank the referees for their useful observations, which we believe have improved the final presentation.

References

[1] J. Barwise, editor. *Handbook of Mathematical Logic.* North-Holland, 1977.

[2] B. Brock. Defstructure for ACL2, 1997. See URL `http://www.cs.utexas.edu/-users/moore/publications/acl2-papers.html#Utilities`.

[3] K. Devlin. *The Joy of Sets: Fundamentals of Contemporary Set Theory.* Springer-Verlag, second edition, 1992.

[4] P. R. Halmos. *Naive Set Theory.* Van Nostrand, 1960.

[5] T. J. Jech. About the axiom of choice. In J. Barwise, editor, *Handbook of Mathematical Logic.* North-Holland, 1977.

[6] M. Kaufmann, P. Manolios, and J S. Moore, editors. *Computer-Aided Reasoning: ACL2 Case Studies.* Kluwer Academic Publishers, June 2000.

[7] M. Kaufmann, P. Manolios, and J S. Moore. *Computer-Aided Reasoning: An Approach.* Kluwer Academic Publishers, July 2000.

[8] M. Kaufmann and J S. Moore. ACL2 homepage. See URL `http://www.cs.-utexas.edu/users/moore/acl2`.

[9] M. Kaufmann and J S. Moore. Structured theory development for a mechanized logic. *Journal of Automated Reasoning*, 26(2):161–203, February 2001.

[10] M. Kaufmann and R. Sumners. Efficient rewriting of data structures in ACL2. In *Proc. ACL2 Workshop 2002*, 2002. See http://www.cs.utexas.edu/users/moore/acl2/workshop-2002/.

[11] K. Kunen. *Set Theory - an Introduction to Independence Proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics.* North-Holland, Amsterdam, 1980.

[12] J S. Moore. Finite set theory in ACL2. In R. J. Boulton and P. B. Jackson, editors, *The 14th International Conference on Theorem Proving in Higher Order Logics (TPHOLs 2001)*, volume 2152 of *LNCS*, pages 313–328. Springer-Verlag, Sept. 2001.

[13] J. Sawada. *Formal Verification of an Advanced Pipelined Machine.* PhD thesis, University of Texas at Austin, Dec. 1999. See URL `http://www.cs.utexas.edu/-users/sawada/dissertation/`.

[14] R. Sumners. An incremental stuttering refinement proof of a concurrent program in ACL2. The University of Texas at Austin, Technical Report TR-00-29, November 2000.