CS313K: Logic, Sets, and Functions

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Lecture 24 – Chap 8 (8.1, 8.2)

Subset in Lists

```
(defun subeq (x y)
  (if (endp x)
      t
      (and (mem (first x) y)
           (subeq (rest x) y))))
```

(subeq '(c a) '(a b c d)) $\Rightarrow$ t

(subeq '(c g) '(a b c d)) $\Rightarrow$ nil

Goal

[$\forall$ `e : (mem e x)` $\longrightarrow$ `(mem e y)]` $\longrightarrow$ `(subeq x y)`

Proof: Induct on `x`.

Base: easy. If `x` is empty, `(subseq x y)` is `T`.

Induction Step

$(\neg(\texttt{endp x})) \wedge$

$([\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})]$

$\longrightarrow (\texttt{subeq (rest x) y}))$

$\longrightarrow$

$[\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})] \longrightarrow (\texttt{subeq x y})$

Induction Step

$$
\begin{array}{l}
(\neg(\texttt{endp x})) \wedge \\
([\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})] \\
\quad \longrightarrow (\texttt{subeq (rest x) y})) \\
\wedge \\
[\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})] \\
\longrightarrow \\
(\texttt{subeq x y})
\end{array}
$$

Induction Step

$$
\begin{array}{l}
(\neg(\texttt{endp x})) \wedge \\
([\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})] \\
\quad \longrightarrow (\texttt{subeq (rest x) y})) \\
\wedge \\
[\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})] \\
\longrightarrow \\
(\texttt{subeq x y})
\end{array}
$$

Case split on whether

$[\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})]$

is true or not.

Induction Step - Case 1

$$
\begin{array}{l}
\quad (\neg(\texttt{endp x})) \wedge \\
\quad \neg[\forall\ \texttt{e : (mem e (rest x))} \longrightarrow \texttt{(mem e y)}] \\
\quad \wedge \\
\quad [\forall\ \texttt{e : (mem e x)} \longrightarrow \texttt{(mem e y)}] \\
\longrightarrow \\
\quad \texttt{(subeq x y)}
\end{array}
$$

Induction Step - Case 1

$$(\neg(\texttt{endp x})) \wedge$$

$$(\exists\ \texttt{e} : \neg((\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})))$$

$$\wedge$$

$$[\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})]$$

$$\longrightarrow$$

$$(\texttt{subeq x y})$$

Induction Step - Case 1

$$(\neg(\texttt{endp x})) \wedge$$

$$(\exists \ \texttt{e} : (\texttt{mem e (rest x)}) \wedge \neg(\texttt{mem e y}))$$

$$\wedge$$

$$[\forall \ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})]$$

$$\longrightarrow$$

$$(\texttt{subeq x y})$$

Induction Step - Case 1

$(\neg(\texttt{endp x})) \wedge$

$(\exists\ \texttt{e : (mem e (rest x))} \wedge \neg(\texttt{mem e y}))$

$\wedge$

$[\forall\ \texttt{e : (mem e x)} \longrightarrow \texttt{(mem e y)}]$

$\longrightarrow$

$\texttt{(subeq x y)}$

Induction Step - Case 1

$(\neg(\texttt{endp x})) \wedge$

$(\texttt{mem e (rest x)}) \wedge \neg(\texttt{mem e y})$

$\wedge$

$[\forall\ \texttt{e : (mem e x)} \longrightarrow \texttt{(mem e y)}]$

$\longrightarrow$

$(\texttt{subeq x y})$

Induction Step - Case 1

$(\neg$(endp x)$)\ \wedge$

(mem e (rest x)) $\wedge\ \neg$(mem e y)

$\wedge$

((mem e x) $\longrightarrow$ (mem e y))

$\longrightarrow$

(subeq x y)

Induction Step - Case 1

$$(\neg(\texttt{endp x})) \wedge$$

$$(\texttt{mem e (rest x)}) \wedge \neg(\texttt{mem e y})$$

$$\wedge$$

$$((\texttt{e=(first x)} \vee (\texttt{mem e (rest x)})) \longrightarrow (\texttt{mem e y}$$

$$\longrightarrow$$

$$(\texttt{subeq x y})$$

Induction Step - Case 1

$$(\neg(\texttt{endp x})) \wedge$$
$$(\texttt{mem e (rest x)}) \wedge \neg(\texttt{mem e y})$$
$$\wedge$$
$$(\texttt{e=(first x)} \vee \texttt{t}) \rightarrow (\texttt{mem e y}))$$
$$\longrightarrow$$
$$(\texttt{subeq x y})$$

Induction Step - Case 1

$(\neg(\texttt{endp x})) \wedge$

$(\texttt{mem e (rest x)}) \wedge \neg(\texttt{mem e y})$

$\wedge$

$(\texttt{mem e y})$

$\longrightarrow$

$(\texttt{subeq x y})$

Obvious!

Induction Step - Case 2

$(\neg(\texttt{endp x})) \wedge$

$[\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})]$

$\wedge$

$(\texttt{subeq (rest x) y}))$

$\wedge$

$[\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})]$

$\longrightarrow$

$(\texttt{subeq x y})$

Induction Step - Case 2

$$
\begin{array}{l}
\quad (\neg(\texttt{endp x})) \land \\
\quad [\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})] \\
\quad \land \\
\quad (\texttt{subeq (rest x) y)}) \\
\quad \land \\
\quad [\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})] \\
\longrightarrow \\
\quad ((\texttt{mem (first x) y}) \land (\texttt{subeq (rest x) y}))
\end{array}
$$

Induction Step - Case 2

$$
\begin{array}{l}
\quad (\neg(\texttt{endp x})) \wedge \\
\quad [\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})] \\
\quad \wedge \\
\quad (\texttt{subeq (rest x) y})) \\
\quad \wedge \\
\quad [\forall\ \texttt{e} : (\texttt{mem e x}) \longrightarrow (\texttt{mem e y})] \\
\longrightarrow \\
\quad ((\texttt{mem (first x) y}) \wedge \texttt{t})
\end{array}
$$

Induction Step - Case 2

$(\neg$(endp x)) $\wedge$

[$\forall$ e : (mem e (rest x)) $\longrightarrow$ (mem e y)]

$\wedge$

(subeq (rest x) y))

$\wedge$

[$\forall$ e : (mem e x) $\longrightarrow$ (mem e y)]

$\longrightarrow$

(mem (first x) y)

Induction Step - Case 2

$(\neg(\texttt{endp x})) \wedge$

$[\forall\ \texttt{e} : (\texttt{mem e (rest x)}) \longrightarrow (\texttt{mem e y})]$

$\wedge$

$(\texttt{subeq (rest x) y}))$

$\wedge$

$((\texttt{mem (first x) x}) \longrightarrow (\texttt{mem (first x) y}))$

$\longrightarrow$

$(\texttt{mem (first x) y})$

Obvious! $\square$

Example "Calculation"

$\mathsf{e} \in \wp(\ \mathtt{A} \cup \mathtt{B}\ )$

$\leftrightarrow$

$\mathsf{e} \in \{\ \mathtt{v} : \mathtt{v} \subseteq (\mathtt{A} \cup \mathtt{B})\ \}$

$\leftrightarrow$

$\mathsf{e} \subseteq (\mathtt{A} \cup \mathtt{B})$

$\leftrightarrow$

$(\forall\ \mathtt{v} : (\mathtt{v} \in \mathsf{e}) \longrightarrow (\mathtt{v} \in (\mathtt{A} \cup \mathtt{B})))$

$\leftrightarrow$

$(\forall\ \mathtt{v} : (\mathtt{v} \in \mathsf{e}) \longrightarrow ((\mathtt{v} \in \mathtt{A}) \vee (\mathtt{v} \in \mathtt{B})))$

Example Proofs

Theorem: $(A \subseteq (A \cup B))$

$\leftrightarrow (\forall v : (v \in A) \rightarrow (v \in (A \cup B)))$

$\leftrightarrow (\forall v : (v \in A) \rightarrow ((v \in A) \vee (v \in B)))$

$\leftrightarrow (\forall v : True) \leftrightarrow True \ \Box$

Theorem: $(A \in \wp(A \cup B))$

$\leftrightarrow (A \in \{v : v \subseteq (A \cup B))$

$\leftrightarrow (A \subseteq (A \cup B)) \leftrightarrow True \ \Box$