

(Problem 12) There is a result for context-free languages analogous to the pumping lemma for regular sets. Suppose that $L(G)$ is the language recognized by a context-free grammar G . This result states that there is a constant N such that if z is a word in $L(G)$ with $\mathcal{L}(z) \geq N$, then z can be written as $uvwxy$ where $\mathcal{L}(uwy) \leq N$, $\mathcal{L}(vx) \geq 1$, and w^iwx^iy belongs to $L(G)$ for $i = 1, 2, 3, \dots$

Use this result to show that there is no context-free grammar G with $L(G) = \{0^n1^n2^n | n = 1, 2, 3, \dots\}$.

Let n any number such that $3n > N$. With this n , $\mathcal{L}(0^n1^n2^n) > N$, so we know that, by this lemma, that $0^n1^n2^n$ can be seen as $uvwxy$ such that $\mathcal{L}(vx) \geq 1$ and w^iwx^iy belongs to $L(G)$. Let's consider $i = 2$ and let's see if there is a m such that $0^m1^m2^m = uvvwxxy$.

if v is not a sequence of the same element, vv will have each element repeated at least twice but not adjacently. For instance, if $v = 01$, then $vv = 0101$. The strings of this kind cannot be a substring of $0^m1^m2^m$, so v can only be a sequence of the same element. The same reasoning can be done on x , obtaining the same result.

$uvvwxxy$ must have exactly the same number of 0s, 1s and 2s. We know that $uvwxy$ has this property, so the same property must apply to vx . But we have seen that each of v and x can have at most one kind of element, so vx cannot contain all three elements. The only case in which all these conditions are respected is when both v and x are empty strings, but $\mathcal{L}(vx) \geq 1$. Consequently $uvvwxxy$ is not a string accepted by the grammar.